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CHAPTERS

ON THE

MODERN GEOMETRY

OF THE

POINT, LINE, AND CIRCLE;

BEING THE SUBSTANCE OF

LECTURES DELIVERED IN THE UNIVERSITY OF DUBLIN TO THE CANDIDATES FOR HONORS OF THE FIRST YEAR IN ARTS.

BY THE

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PREFACE.

The work now offered to the public contains, as its title indicates, the substance of lectures delivered for some years in the University of Dublin to the candidates for mathematical honors of the first year in arts; and supposes, accordingly, a previous acquaintance only with the first six books of the Elements of Euclid, and with just that amount of the principles of Elementary Algebra essential to an intelligent conception of the nature of signs, and of the meaning and use of the ordinary symbols of operation and quantity.

The acknowledged want of a systematic treatise on Modern Elementary Geometry, adapted to the requirements of students unacquainted with the higher processes of Algebraic Analysis, which of late years have been applied so successfully to the extension of geometrical knowledge, has induced the author to come forward with the present attempt to supply the deficiency. The only existing work of the same nature in the English language with which he is acquainted, the "Principles of Modern Geometry," of the late lamented Dr. Mulcahy,

published in the year 1852, being now confessedly behind the present state of the subject; and the only other work of the same nature in any language with which he is acquainted, the elaborate and masterly "Traité de Géométrie Supérieure" of the justly celebrated M. Chasles, published in the same year, having become so scarce as to be now hardly attainable at any price.

Though designed mainly for the instruction of students of the comparatively limited mathematical knowledge generally possessed at the transition from school to university life, and arranged with special reference to the existing course of mathematical instruction in the University of Dublin, the author has spared no pains to render the work as generally interesting and instructive as the extent of his subject admitted. The order adopted, though framed on the basis of an existing arrangement, appeared as natural as any other he could have substituted for it; the principles established have been considered in all the generality, and stated with all the freedom from ambiguity, of which they appeared to him susceptible; and the demonstrations submitted, which are to a considerable extent original, have been presented as directly referred to ultimate principles, and as completely disencumbered of unessential details, as he was capable of rendering them.

To the second of the works above referred to, the "Traité de Géométrie Supérieure" of M. Chasles, the author is indebted for many important suggestions in the advanced chapters of the work; in those especially on the Theories of Anharmonic Section, Homographic Division, Involution, &c., of which its illustrious author was virtually the originator as well as the nomenclator, it will be at once seen that he has profited largely by the results so ably developed in the corresponding chapters of that elaborate work, while at the same time he can in no sense be regarded as the mere copyist of any of its contents.

To the Board of Trinity College the best thanks of the author are due for the liberal assistance they have given towards defraying the expenses of the work.

Trinity College, Dublin,
October, 1863.

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THE MODERN GEOMETRY OF THE POINT, LINE, AND CIRCLE.

CHAPTER I.

ON THE DOUBLE ACCEPTATION OF GEOMETRICAL TERMS.

1. GEOMETRICAL propositions refer either to the comparative *magnitudes* of geometrical *quantities*, as in the propositions: "Rectangle under sum and difference = difference of squares," "Square of sum + square of difference = twice sum of squares," &c., or to the relative *positions* of geometrical *figures*, as in the propositions: "All points equidistant from the same point lie on the same circle," "All lines equidistant from the same point touch the same circle," &c. Hence the modern division of the science of Geometry into the two departments of *Geometry of magnitude or quantity* and *Geometry of position or figure* respectively.

2. The ordinary terms of Geometry are, with few exceptions, employed in double acceptations with reference to these two departments, and denote sometimes *magnitudes* and sometimes *figures;* the familiar term "line," for instance, denoting sometimes the indefinite figure so denominated extending to infinity in both directions and sometimes the distance from one point to another; the equally familiar term "angle," again denoting sometimes the complete figure formed by two indefinite lines extending to infinity in both directions and sometimes the inclination of one line to another. The ambiguity

arising from this duality of application rarely, if ever, causes any inconvenience or confusion, as the sense in which geometrical terms are employed is generally apparent from the context in which they occur, as for instance in the expressions: "points of bisection of a line," "lines of bisection of an angle," &c.

3. The literal symbols also by which geometrical figures of all kinds are wont to be represented, are employed occasionally in a similar duality of application with reference to the two departments of geometry; thus if A and B represent two points, AB represents indifferently the indefinite line passing through both and the linear interval between them. If A and B represent two lines, AB represents indifferently the unique point common to both and the angular interval between them. If A and B represent one a point and the other a line, AB represents indifferently the indefinite line passing through the former at right angles to the latter, and the perpendicular interval between them; for the reason already stated the ambiguity arising from this duality of application rarely, if ever, causes any inconvenience or confusion in practice.

4. Of the two different ways in which linear and angular magnitudes are alike ordinarily represented, viz. by the two letters which represent their extreme points or lines, or by a single letter denoting the number of linear or angular units they contain; the latter or *uniliteral* notation is generally the more convenient when *magnitude* only need be attended to, as in the familiar instance of the triangle in which the three sides are ordinarily represented by the three small letters a, b, c, and the three respectively opposite angles by the three corresponding capitals A, B, C, a notation than which nothing could be more convenient; but the former or *biliteral* notation is, on the contrary, the more convenient, when, as is often the case, *direction* as well as magnitude has to be taken into account, which under the biliteral notation may be indicated, in a manner at once simple and expressive, merely by the *order* in which the two letters are written, AB naturally representing the segment, or the angle, or the perpendicular

intercepted between the two points, or the two lines, or the point and line, A and B considered as measured in the direction from A to B, and BA the same segment, or angle, or perpendicular considered as measured in the opposite direction from B to A; this mode of distinction we shall have frequent occasion to employ in the sequel.

5. When a geometrical magnitude of any kind is represented or said to be represented, as it often is, by a *number*, or by a letter regarded as the representative of a number, it is always to be remembered that what is meant by such number or representative letter is the ratio the magnitude bears to some other magnitude of the same kind, given or assumed arbitrarily, but not either evanescent or infinite, to which it is implicitly, if not expressly, referred as a standard, and which is called the *unit* of that particular kind of magnitude, because that when the compared and standard magnitudes are equal, the number representing the former is then unity. The given or assumed unit of any particular kind of magnitude may have theoretically *any* finite value, as, whatever it is, it always disappears whenever different magnitudes of the same kind are compared with each other, their relative magnitudes, or ratios to each other, being of course independent of the arbitrary standard by which their absolute values may happen to be estimated; it is thus, and thus only, that magnitudes other than abstract numbers become subjects of *calculation*, the proper and only subjects of which are *numbers* and numbers alone.

6. With respect to the three species of geometrical magnitude, *length*, *area*, and *volume*, it is to be observed that as the magnitudes themselves are not all independent of each other, but on the contrary vary simultaneously according to known laws, their three units consequently are never *all* arbitrary *together*, but are always made to correspond to each other according to the same laws of simultaneous variation in a manner at once obvious and natural; areas and volumes varying, cæteris paribus, as the squares and as the cubes respectively of the lengths on which they depend, the unit of area

accordingly is always the square and the unit of volume the cube of the unit of length; the latter, however, or more generally some one of the three, being arbitrary; it is for this reason that we are justified in asserting the area of a parallelogram and the volume of a parallelopiped to be equal in abstract numbers to the products of their two and of their three dimensions respectively, and similarly of other areas and volumes as having the same necessary and known connection with the lengths on which they depend.

7. With respect to the only remaining species of geometrical magnitude, viz. *inclination*, as no connection exists between it and any of the other three, its unit is therefore at once arbitrary and independent of any of theirs; any finite angle, consequently, may be given or assumed at pleasure, considered as the angular unit, and all other angles estimated by the numbers, integer or fractional, of such units contained in them; and this accordingly is what is done in Astronomy, Geography, Navigation, Geodesy, &c., and in other practical applications of Geometry where angles are ordinarily estimated by the numbers of *degrees*, *minutes*, and *seconds*, &c. which they contain.

Theoretically considered, the most convenient unit of angular measure as well in Geometry as in the science which treats more especially of angles and their relations, is *the angle which from the centre of a circle subtends an arc = the radius*, and which, as all circles are similar figures, is consequently unique, because in reference to it as unit the numerical value of any angle is simply *the ratio of the subtending arc to the radius* in any circle described round the vertex as centre, a value simpler than for any other unit. Practically considered, however, this unit has the twofold disadvantage; firstly, of being so large that angles of ordinary magnitude, if referred to it, must be expressed as fractions; and, secondly, of not being a sub-multiple of, or even commensurable with, four right angles, the exact divisions and sub-divisions of which are of such importance in all practical subjects.

8. As in Arithmetic the third proportional to any number and unity is termed the *reciprocal* of the number, so in Geo-

metry the third proportional to any magnitude and the unit, whatever it be, to which it is referred, is termed the *reciprocal* of the magnitude.

By taking the reciprocal of the reciprocal, as thus defined, either of a magnitude or number, we evidently get back again the original magnitude or number. Hence the reason why magnitudes or numbers so related are termed *reciprocals* to each other, the process by which either produces the other always reciprocally reproducing itself from the other.

The product of the extremes being equal to the square of the mean in every proportion of three terms, the product of every pair of numbers reciprocals to each other $= 1$, and that of every pair of magnitudes of any kind reciprocals to each other = the square of the common unit, whatever it be, to which they are referred; and, conversely, if the product of two numbers $= 1$, or the product of two magnitudes of any kind = the square of the unit to which they are referred, such numbers or magnitudes are reciprocals to each other.

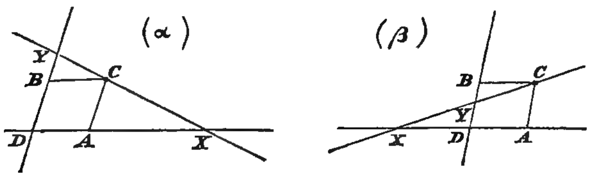
When two ratios $a : b$ and $c : d$ are reciprocals to each other, the four component magnitudes a, b, c, d, whatever be their nature, are evidently "reciprocally proportional" in Euclid's meaning of the phrase. (Euc. VI. 14, 15, 16).

9. As in Arithmetic the numbers *nothing* and *infinity* are reciprocals to each other, each being evidently the third proportional to the other and *any finite number*, so in Geometry *evanescent* and *infinite* values of any kind of magnitude are always reciprocals to each other, *whatever be the absolute value of the unit to which they are referred*, each being evidently the third proportional to the other and *any finite value* of the same kind of magnitude.

The rectangle under two linear magnitudes, reciprocals to each other, being constant and = the square of the unit, whatever it be, to which they are referred. The reader, familiar with the Second Book of Euclid, may take as exercises in its principles the four following problems: "*Given the sum, difference, sum of squares, or, difference of squares, of two linear magnitudes reciprocals to each other to a given unit, to determine the magnitudes.*"

10. There are several constructions by which pairs of reciprocals in linear magnitudes may be simultaneously determined, of which the following is perhaps the simplest:

Round any one of the four corners C of any equilateral parallelogram $ABCD$ the common length of whose four sides



= the linear unit, let an indefinite line XY be conceived to revolve intersecting the two sides AD and BD opposite to C in two variable points X and Y; the intercepts AX and BY between the two points of meeting and the two corners A and B adjacent to C are always reciprocals to each other.

For, by similar triangles XAC and CBY,

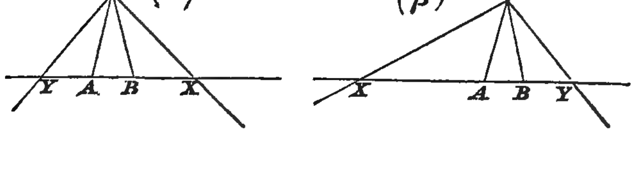
$$AX : AC = BC : BY \text{ or } AX.BY = AC.BC$$

in every position of the revolving line, and therefore, &c.

The parallelogram in the above need not be equilateral; any parallelogram, the rectangle under whose adjacent sides $CA.CB =$ the square of the linear unit, would obviously do as well.

11. In case it should be desirable to have the simultaneous reciprocals AX and BY measured on the same in place of on different lines, the following modification of the above may be employed for the purpose:

Round the vertex C of any isosceles triangle ACB, the



common length of whose sides = the linear unit, let two indefinite lines CX and CY inclined to each other at a constant angle equal to either base angle of the triangle be conceived to revolve intersecting the base AB in two variable points X and Y; the intercepts AX and BY between the two points of meeting and the two extremities of the base AB, for which the three angles CAX, CBY, and XCY are of the same affection; that is, all three acute (fig. α) or all three obtuse (fig. β) are always reciprocals to each other.

This is obviously identical with the preceding construction modified by turning the unit line CB round C, bringing with it the two indefinite lines BY and CY until the former coincides with AX, and the same demonstration, word for word, and letter for letter, applies indifferently to either.

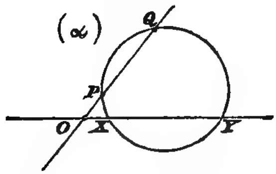
Since during the revolution of the constant angle XCY its acute and obtuse regions alternately comprehend the intercept XY, should any doubt exist in any particular position as to how the two points X and Y correspond to the two A and B in measuring the reciprocals AX and BY, it will be at once settled by remembering, as above stated, (see the figures of the original as well as of the modified construction which have been drawn to correspond) that the angles CAX and CBY must be always of the same affection with XCY.

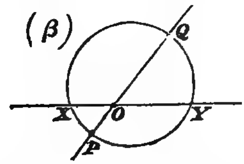
If the vertical angle of the isosceles triangle ACB were nothing, its unit sides CA and CB would coincide and be perpendicular to XY; the constant revolving angle XCY would be right in every position; the two reciprocals AX and BY would be measured from a common origin, and the ambiguity adverted to above would not exist: in the corresponding case of the original construction the lozenge $ABCD$ would evidently be a square.

12. The following, however, is the most convenient construction for the simultaneous determination of pairs of conterminous reciprocals upon any given indefinite line MN, inasmuch as by it they may be determined at pleasure either in similar or in opposite directions from any given common origin O.

Drawing arbitrarily from the common origin O in any similar or opposite directions, according as the directions of the

reciprocals are to be similar or opposite, any two lengths OP





and OQ, the rectangle under which $OP.OQ=$ the square of the linear unit. Every circle passing through their two extremities P and Q intersects the given line MN in two points X and Y whose distances from O are always reciprocals to each other.

For, Euc. III. 35, 36, $OX.OY=OP.OQ$, whatever be the circle, and therefore, &c.

The above three methods have all the common advantage of allowing to both reciprocals every range of magnitude from nothing to infinity, and of shewing very clearly how the passage of either through nothing or infinity is accompanied by the simultaneous passage of the other through infinity or nothing, whatever, in any case, be the absolute value of the unit to which they are referred, provided only it be finite.

13. Geometrical magnitudes of every kind, when compared with others of the same kind, present in their evanescent and infinite states some anomalous peculiarities, to which, as constantly occurring in geometrical investigation, we proceed to call early attention.

The product of an evanescent or of an infinite with any finite magnitude and the ratio of an evanescent or of an infinite to any finite magnitude being necessarily evanescent or infinite, when therefore two geometrical magnitudes of any kind have *any finite* product or ratio, one necessarily becomes infinite as the other vanishes, and conversely, in the former case, and both vanish or become infinite together in the latter case; hence, as in abstract numbers the product of 0 with ∞ or of ∞ with 0, and the ratio of 0 to 0 or of ∞ to ∞ is plainly *indeterminate*, so in geometrical magnitudes of every

kind, the product of an evanescent with an infinite or of an infinite with an evanescent magnitude, and the ratio of an evanescent to an evanescent or of an infinite to an infinite magnitude, *considered in the abstract*, is also indeterminate; though in every particular instance in which either product or ratio actually arises it has generally some particular definite value determinable and to be determined from consideration of the particular circumstances under which it arises; as, for instance, if the product or ratio were *constant* in the general and therefore in every particular state of the magnitudes.

14. The ratio of two magnitudes of any kind, considered in the abstract, being thus indeterminate when the magnitudes are both either evanescent or infinite, it follows therefore that the two *criteria of equality* between two magnitudes of the same kind when compared with each other, viz. that 1°. their ratio $=1$, and 2°. their difference $=0$, each of which necessarily involves the other so long as the magnitudes are finite, do not involve each other when the magnitudes are either evanescent or infinite, for while the difference between two evanescent magnitudes is always $=0$, their ratio, as above shewn, may have any value $=$ or not $=1$, and while the ratio of two infinite magnitudes may be and often is $=1$, their difference, as may be easily shewn, may have any value $=$ or not $=0$.

15. The following useful example may be taken as an illustration of the preceding observation:

The ratio of the distances of a point P at infinity from any two points A and B not at infinity is always equal to unity, though their difference may (Euc. I. 20) *have any value from nothing to the interval AB.*

For, whatever be the position of P, whether at or not at infinity, or on or not on the line AB, since (Euc. I. 20) PA differs from PB by a quantity not exceeding AB, therefore $PA : PB$ differs from $PB : PB$, or 1, by a quantity not exceeding $AB : PB$, which quantity $=0$, whatever be the length of AB whether evanescent or finite, when $PB = \infty$, that is, when P is any where at infinity whether on or not on the line AB.

In the particular case when the two points A and B coincide, then for every position of P, whether at infinity or not, the two criteria of equality $PA : PB = 1$, and $PA \sim PB = 0$ evidently hold, except only for the point $A = B$ itself, for which the ratio assumes the indeterminate form $0 : 0$, and, therefore, (13) may have any value as well as 1. This particular case often occurs in geometrical investigations, and whenever it does its peculiarity must always be attended to.

In the general case when A and B do not coincide, for every point P on the indefinite line bisecting internally at right angles the interval AB, whether at infinity or not, both criteria of equality $PA : PB = 1$, and $PA \sim PB = 0$, hold without any exception, while for a point P not on that line $PA \sim PB$ is never $= 0$, and $PA : PB$ is therefore $= 1$ only when P is at infinity.

In the general case again, for every point P on the indefinite line AB itself, whether at infinity or not, $PA \sim PB$, except only for the finite interval between A and B, has (Euc. I. 20) the greatest possible value AB, and therefore for points external to that interval $PA : PB = 1$ only when P is at infinity, in which position it is consequently termed *the point of external bisection* of the segment AB. Hence we see that—

The point of external bisection of any finite segment of a line is the point at which the line intersects infinity, and conversely, the point at which a line intersects infinity is the point of external bisection of any finite segment of the line.

In the particular case when the segment is evanescent, then, as already stated, every point on the line, except only that at which the extremities coincide, is indifferently a point of external bisection of the segment.

16. Admitting that any number of lines passing through a common point divide similarly (Euc. VI. 10) any two parallel lines in the ratio of their distances from the point, and that, conversely, any number of lines dividing any two parallel lines similarly in any ratio pass through a common point whose distances from the parallels are in that ratio; the following very important, but at first sight somewhat paradoxical, con-

clusion respecting points at infinity, results immediately from the general property of the preceding article, viz.—

Every system of lines passing through a common point at infinity is a system of parallel lines; and conversely, every system of parallel lines is a system of lines passing through a common point at infinity.

For, conceiving any two parallel lines L and L' drawn arbitrarily intersecting the entire system of lines, in either case, in two systems of points A, B, C, D, &c. and A', B', C', D', &c., then since, in the former case, the several lines AA', BB', CC', DD', &c. pass, by hypothesis, through a common point O, therefore, (Euc. VI. 4)

$$AB : A'B' = AC : A'C' = AD : A'D', \text{\&c.} = AO : A'O, = 1,$$

since, by hypothesis, O is at infinity (15); therefore $AB = A'B'$, $AC = A'C'$, $AD = A'D'$, &c.; and therefore (Euc. I. 33) BB', CC', DD', &c. are all parallel to AA' and to each other; and since, in the latter case, the several lines AA', BB', CC', DD', &c. are, by hypothesis, parallel; therefore (Euc. I. 34) $AB = A'B'$, $AC = A'C'$, $AD = A'D'$, &c.; and, therefore, as $AB : A'B' = AC : A'C' = AD : A'D'$, &c., the several lines BB', CC', DD', &c. all intersect AA' at the same point O (Euc. VI. 4); and as the common ratio $= 1$ that point O is at infinity (15).

The above is but one of a multitude of arguments for the truth of a conclusion long placed beyond all question by the simplest considerations of projection and perspective.

By a very slight modification Euclid's excellent definition of parallel lines, those, viz., "which lying in the same plane never meet though indefinitely produced," might be made to express the preceding most important and indeed fundamental property of such lines without failing to convey at the same time the notion intended by the original. The simple substitution of the two words *until infinitely* in place of the two *though indefinitely* would manifestly effect this.

It is evident from the above that the *position* of a point at infinity both determines and is determined by the *direction* of any line passing through it.

17. If a variable line be conceived to revolve continuously in one direction round a fixed point and to intersect in every position a fixed line not passing through the point; the point of intersection evidently traverses continuously in one direction the entire fixed line in the course of each complete semi-revolution of the variable line; approaches to infinity in the direction of its motion as the latter approaches to a position of parallelism with the former; reaches infinity as that position is attained; and emerges again from infinity from the opposite direction when that position is passed; from this and from many other considerations geometers have long satisfied themselves that

The two opposite directions of every line, not itself at infinity, are to be regarded, not as reaching infinity at two different and opposite points, but as running into each other and meeting at a single point at infinity.

Hence the propriety of the expression "point of external bisection" of any finite segment of a line (15).

Paradoxical as the above conclusion may appear when first stated, the grounds confirmatory of it are so numerous and varied that any early hesitation in admitting its legitimacy is generally very rapidly got over.

18. If the centre of a variable circle touching a fixed line at a fixed point be conceived to traverse continuously in one direction the entire circuit (17) of the orthogonal line passing through the point, starting from and returning to the point through infinity (Euc. III. 19). The circle itself evidently commences from evanescence with the commencement of the motion; expands continuously at the side of the line corresponding to its direction during the first half of the circuit; opens out into the line itself as infinity is reached; contracts continuously at the opposite side of the line during the second half of the circuit; and, terminates in evanesence with the completion of the motion. Hence, and from many other considerations, it appears that—

Every point not at infinity may be regarded as a circle of evanescent radius whose centre is the point; and every line not at infinity as a circle of infinite radius whose centre is the point at infinity in the direction orthogonal to the line (16).

In the geometry of the point, line, and circle, therefore the point and line are the limiting forms of the circle in the extreme cases of its radius being evanescent and infinite.

19. If a variable line be considered to revolve continuously in one direction round a fixed point, and to intersect in every position a fixed circle passing through the point, the variable point of intersection evidently traverses continuously in one direction the entire circumference of the circle in the course of each complete semi-revolution of the line; and, on its way every time, approaches to, reaches, and passes through the fixed point as the line approaches to, reaches, and passes through the particular position *in which it is a tangent to the circle at that point.* Hence, and from innumerable other considerations, it appears that—

When the two points of intersection of a line and circle coincide, the line and circle touch at the point of coincidence.

And generally that—

When two points of intersection of any two figures coincide, the figures themselves touch at the point of coincidence.

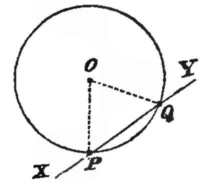
This, indeed, as fundamentally correct in conception and invariably simple in application, might be made the formal *criterion of contact* in elementary, as it is in advanced, geometry; and from it the several known properties respecting the contact of circles with lines and with each other, established in the Third Book of Euclid and elsewhere, might be easily shewn to be mere corollaries from more general properties respecting their intersection, deduced by simply introducing into the latter the particular supposition of coincidence between their two, in general separate, points of intersection. A few examples will shew this more clearly.

Ex. 1°. ***A line and circle or two circles having contact at any point can never meet again either by contact or intersection.*** (Euc. III. 13 and 16.)

For they can never under any circumstances meet at all in *more* than *two* points (Euc. III. 2 and 10); which property being true *in general,* whatever be the interval between the points, is *therefore* true *in the particular case* where the interval $=0$; that is, when the figures touch.

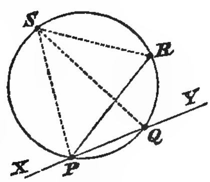
Ex. 2°. ***At every point on a circle the tangent is perpendicular to the radius.*** (Euc. III. 18.)

Let P be the point, XY *any* line passing through it, Q the other point in which XY meets the circle, and OP and OQ the radii to P and Q; then, *whatever be the interval* PQ, since the triangle POQ is always isosceles, the two external angles OPX and OQY are *always* equal (Euc. I. 5); they are *therefore* equal *in the particular case* when Q coincides with P, and therefore OQ with OP, and therefore the angle OQY with the angle OPY, in *that case*, therefore, the angles OPX and OPY are equal; and, therefore, (Euc. I. def. 11) the radius OP is perpendicular to the *tangent* XY.

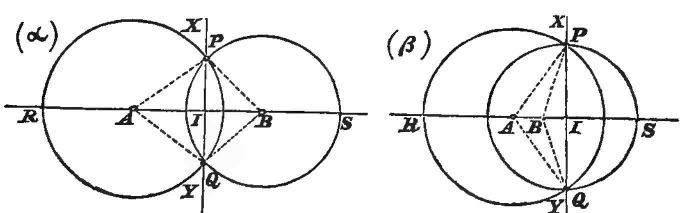


Ex. 3°. *At every point on a circle the angles made by any chord with the tangent are equal to the angles in the alternate segments,* (Euc. III. 32).

Let P be the point, PR the chord, XY *any* line passing through P, Q the other point in which XY meets the circle, S any arbitrary point on the circle, and SP, SQ, SR the lines connecting it with P, Q, R, then, *whatever be the interval* PQ, the angles RPY and RSQ being in the same segment are *always* equal (Euc. III. 21); they are *therefore* equal *in the particular case* when Q coincides with P, and therefore SQ with SP, and therefore the angle RSQ with the angle RSP, in *that* case therefore the angles RPY and RSP are equal, that is, the angle the chord PR makes with the *tangent* XY is equal to the angle in the alternate segment PSR.



Ex. 4°. *When two circles touch, externally or internally, the line joining their centres passes through the point of contact and is perpendicular to the line touching both at that point,* (Euc. III. 11 and 12).



Let PQR and PQS be *any* two *intersecting* circles, P and Q their two points of intersection, A and B their two centres, and XY the indefinite line passing through P and Q; then, on account of the two isosceles triangles PAQ and PBQ formed by connecting A and B with P and Q, the line AB connecting their vertices A and B always both bisects and

is perpendicular to their common base PQ (Euc. III. 3); and this being *always* true in general, whatever be the length of PQ, is *therefore* true *in the particular case* when that length $=0$, that is, when the two points P and Q coincide, but when they do, their middle point I coincides with both, the two circles touch, externally or internally, at the point of coincidence, and the indefinite line XY touches both at that point.

In every application of the above method, one precaution, observed it will be perceived in each of the above illustrative examples, is invariably to be attended to. The supposition of coincidence between the two points of intersection P and Q, in which the contact of the figures consists, is *never* to be introduced *until* the more general property, independent of the distance between them, has *first* been established.

20. As in the compound figure consisting of a line and circle variable in relative position with respect to each other, the two points common to both pass evidently from separation, through coincidence, to simultaneous disappearance, or conversely, as the distance of the line from the centre passes from being $<$, through being $=$, to being $>$ the radius of the circle, or conversely; so in the compound figure consisting of a point and circle variable in relative position with respect to each other, the two tangents common to both pass similarly from separation, through coincidence, to simultaneous disappearance, or conversely, as the distance of the point from the centre passes from being $>$, through being $=$, to being $<$ the radius of the circle, or conversely. Hence, as in many ways otherwise, it appears that—

As every tangent to a circle or any other figure is the connector of two coincident points on the circle or figure, and conversely, so every point on a circle or any other figure is the intersection of two coincident tangents to the circle or figure, and conversely.

In the applications of this, as of the preceding principle, of which it is the correlative, the same precaution again is invariably to be observed; in investigating any property of a point on a circle or any other figure regarded as the intersection of two coincident tangents to the circle or figure, the supposition of coincidence between the two tangents is never to be intro-

duced until the more general property of the point of intersection of *any* two tangents, in which it is involved, has first been established.

21. In the language of modern geometry every two points, lines, or other similar elements of, or connected with, any compound figure, which with change of relative position among the constituents of the figure pass or are liable to pass, as above described, from *separation*, through *coincidence*, to *simultaneous disappearance*, or conversely, are termed *contingent* as distinguished from *permanent* elements of the figure, and are said to be *real* or *imaginary* according as they *happen to be* apparent or non-apparent to sense or conception. Geometers of course have not, nor do they profess to have, any conception of the nature of contingent elements in their imaginary state, but they find it preferable, on the grounds both of convenience and accuracy, to regard and speak of them as imaginary rather than as non-existent in that state: in the transition from the real to the imaginary state, and conversely, contingent elements pass invariably through coincidence, through which, as above described, they always change state together.

In the geometry of the point, line, and circle, it is only in figures involving, directly or indirectly, the latter in its finite form, that contingent elements from their nature could occur; in figures, however complicated, consisting of points and lines only all elements not depending on the circle in its finite form are invariably permanent.

22. When a line and figure of any kind intersect, the angles between the line and the tangents to the figure at the several points of intersection are termed the angles of intersection of the line and figure at the points; when two figures of any kind intersect, the angles between the tangents to them at the several points of intersection are termed the angles of intersection of the figures at the points; in the cases of a line and circle and of two circles the angles of intersection at the two points of intersection being evidently equal, each separately is called *the angle of intersection* of the figures.

With respect to the angle of intersection of a line and circle it is evident that:

1°. Every line passing through the centre of a circle intersects the circle at right angles; and conversely, every line intersecting a circle at right angles passes through the centre of the circle, (Euc. III. 18, 19).

2°. Every line dividing a circle into segments containing any angle, intersects the circle at the angle in the segments; and conversely, every line intersecting a circle at any angle divides the circle into segments containing the angle, (Euc. III. 32).

3°. A variable line whose distance from a fixed point is constant intersects at a constant angle every circle of which the point is the centre.

And with respect to the angle of intersection of two circles that:

1′. Every circle touching at either extremity any diameter of another circle intersects the other at right angles; and conversely, every circle intersecting another at right angles touches at each point of intersection a diameter of the other.

2′. Every circle touching at either extremity any chord of another circle intersects the other at the angle in the segments determined by the chord; and conversely, every circle intersecting another at any angle touches at each point of intersection a chord dividing the other into segments containing the angle.

3′. A variable circle of constant radius the distance of whose centre from a fixed point is constant intersects at a constant angle every circle of which the point is the centre.

A line and circle, two circles, or any other two figures, intersecting at right angles, are said *to cut orthogonally*, or, as it is sometimes termed, *to be orthotemic.*

23. In order to avoid the ambiguity as to which of the two supplemental angles, regarded as magnitudes, between the two tangents at either point of intersection of two circles is to be regarded as *the* angle of intersection of the circles,

in cases in which it is necessary, as it often is, to distinguish between them, the following convention has been agreed to by geometers.

The radius being perpendicular to the tangent at every point of a circle, and the two supplemental angles between any two lines being equal to those between any two perpendiculars to them, if from either point of intersection P or Q (fig., Ex. 4°, Art. 19) of the two circles, the two radii, PA and PB, or, QA and QB, be drawn, one of the two supplemental angles between the two tangents is equal to the *internal* and the other to the *external* angle between the two radii; the *former*, APB or AQB, is that which is considered as the angle of intersection of the circles; this is obviously tantamount to regarding that angle as measured either from the convex circumference of one circle to the concave circumference of the other; or, *vice versâ*, from the concave of one to the convex of the other; but not either from the concave of one to the concave of the other, or from the convex of one to the convex of the other.

In accordance with this convention the angle of intersection of two circles is to be regarded as acute, right, or obtuse, according as the square of the distance between their centres A and B is less than, equal to, or greater than the sum of the squares of their radii AP and BP, or AQ and BQ (Euc. II. 12, 13); in the extreme case of the former when $AB =$ the difference of the radii, that is, when the circles touch at the same side of their common tangent, the angle of intersection is to be regarded as $=0$; and in the extreme case of the latter when $AB =$ the sum of the radii, that is, when the circles touch at opposite sides of their common tangent, the angle of intersection is to be regarded as $=$ two right angles; and, for the same reason, generally, when any two figures touch, their angle of intersection at the point of contact is to be regarded as $=0$, or $=$ two right angles, according as they lie at the same side or at opposite sides of their common tangent at the point.

24. In *every* case of the *comparison* of two or more angles regarded as magnitudes (2) it is to be remembered: 1°. That

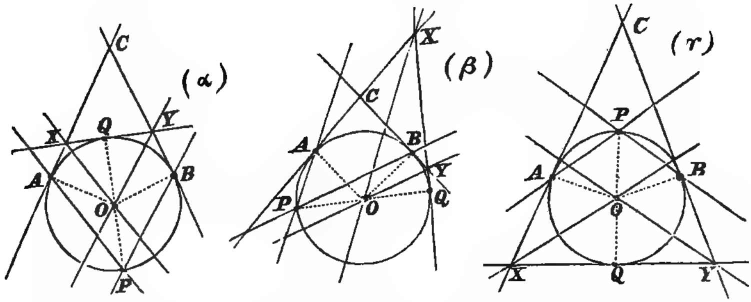
every two finite conterminous lines determine *two* different angular intervals of separation from each other, one exceeding by as much as the other falls short of two right angles, and having in the abstract equal claims to be regarded as *the* angle between the lines; and, 2°. That every two intersecting indefinite lines determine *two* pairs of opposite equiangular regions, one exceeding, in angular interval of separation between the determining lines, by as much as the other falls short of a right angle, and having in the abstract equal claims to be regarded as *the* angle between the lines. The twofold source of ambiguity thus arising must always be attended to in comparing angular magnitudes, as, whatever be the nature of two compared angles, the greater interval for one corresponds often to the lesser for the other in the former case, and the obtuse region for one corresponds often to the acute for the other in the latter case; and that even for angles *similar* as *figures*, that is, whose sides, whether finite or indefinite, are capable of simultaneous coincidence. Whenever, therefore, two angles different in position but similar in form, are said, as they often are, to be *equal*, and when an angle variable in position but invariable in form is said, as it often is, to be *constant*, the terms so employed, though applicable properly to magnitudes only, are to be regarded as indicating the aforesaid similarity or invariability of *form*, rather than absolute equality or constancy of *value*, in such cases generally.

25. The two following examples, of repeated occurrence in the modern geometry of the circle, are important illustrations of the preceding observations.

1°. *A variable point on the circumference of a fixed circle subtends a constant angle at any two fixed points on the circle.*

2°. *The segment of a variable tangent intercepted between any two fixed tangents to a circle subtends a constant angle at the centre of the circle.*

To prove 1°. Let O be the centre of the circle, A and B the two fixed points and P the variable point; the angle APB is, according to the position of P, equal to half the less or greater



angular interval AOB, and therefore constant in the sense above explained.

For, joining OA, OB, OP, and producing the latter through O to meet the circle again at Q; then, as in Euc. III. 21, 22, the angles APO and BPO being the halves of the angles AOQ and BOQ, the sum, or difference as the case may be, of the former, that is the angle APB, = half the sum, or difference, of the latter, that is, half the (less or greater) angle AOB; and therefore &c.

To prove 2°. Let AC and BC be the two fixed tangents, XY the segment of the variable tangent intercepted between them, Q its point of contact, and O, as before, the centre of the circle; the angle XOY is, according to the position of XY, equal to half the less or greater angular interval AOB, and therefore constant in the sense above explained.

For, joining OA, OB, OQ; then, Euc. III. 17, the angles XOQ and YOQ being the halves of the angles AOQ and BOQ, the sum, or difference as the case may be, of the former, that is, the angle XOY = half the sum, or difference, of the latter, that is half the (less or greater) angle AOB; and therefore &c.

Now it is evident that it is as figures and not as magnitudes (2) the two angles APB and XOY are strictly speaking invariable; for as the two points P and Q, on which their positions depend, traverse the entire circumference of the circle, their magnitudes in the positions indicated in fig. (γ), in which they are halves of the greater angular interval AOB, are evidently *the supplements* of their magnitudes in the positions indicated

in the figures (α) and (β), in which they are halves of the lesser angular interval AOB; and so universally in all cases of the same nature, two finite conterminous lines presenting indifferently their greater and lesser angular intervals of separation, and two indefinite intersecting lines their obtuse and acute regions of figure, when revolving through four right angles.

In the particular cases when either the two fixed points A and B or the two variable points P and Q are diametrically opposite points of the circle, the two constant angles APB and XOY are always not only similar as figures but equal as magnitudes; for in the former case, whatever be the positions of P and Q, the two pairs of lines PA and PB, OX and OY intersect evidently at right angles, and therefore &c., and in the latter case (that represented in the figures), whatever be the positions of A and B, the two pairs of lines PA and OX, PB and OY are evidently parallels, and therefore &c.

CHAPTER II.

ON THE DOUBLE GENERATION OF GEOMETRICAL FIGURES.

26. When a variable point moving according to some law lies in every position on a figure of any form, such figure is termed the *locus* of the point. When a variable line moving according to some law touches in every position a figure of any form, such figure is termed the *envelope* of the line. As every simple figure, whatever be its form, may be conceived to be generated, either, if not itself a point, by the continued motion of a point, or, if not itself a line, by the continued motion of a line; with those two exceptions therefore every simple figure in geometry, whether existing alone or in combination with other figures, may be regarded *either as the locus of a variable point or as the envelope of a variable line.*

27. The law directing the movement of the generating point or line being given, the nature of the figure described or enveloped is implicitly given with it, though its actual determination presents of course very different degrees of difficulty in different cases; thus, for instance, the locus of a variable point, or the envelope of a variable line, moving so as to preserve a constant distance from a fixed point, is evidently a circle of which the fixed point and constant distance are the centre and radius.

28. But the law directing the movement of the generating point or line, by which a figure, the nature of which is given, may be described or enveloped, need not necessarily be that expressing the primary or fundamental property by which such figure may have been defined, but on the contrary may be one resulting from *any* of its secondary or derived properties in-

stead: thus, though a circle may, as above, be regarded either as the locus of a variable point, or as the envelope of a variable line, the distance or the square of the distance of which from a fixed point is constant; it may also, as will hereafter appear, be regarded either as the locus of a variable point the sum of the squares of whose distances, or as the envelope of a variable line the sum of whose distances, from any number of fixed points is constant.

29. A single geometrical condition governing the movement of a variable point or line is sufficient in all cases to restrict the point or line to some locus or envelope; thus, for instance, the single condition that a variable point subtend, or that a variable line intersect, a fixed circle at a constant angle, is sufficient to restrict the point or line to a concentric circle as its locus or envelope, of this the reason is evident, for while no condition on the one hand leaves the point or line free to occupy *any* position, two conditions on the other hand suffice when independent to fix it altogether.

30. The locus of a variable point or the envelope of a variable line may be, and often is, a compound figure whose component simple figures satisfy separately the condition governing the movement of the point or line; thus, for instance, the locus of a variable point whose distances from two fixed lines are equal consists evidently of the two lines of bisection external and internal of the angle determined by the lines, and the envelope of a variable line whose distances from two fixed points are equal consists evidently of the two points of bisection external and internal of the segment determined by the points; and similarly for any other constant ratio as well as that of equality. In such cases the compound figure consisting of the two or more simple figures is sometimes termed the *complete* locus or envelope of the point or line.

31. With respect to *particular cases* of loci and envelopes it is to be observed in general that—

1°. A locus or envelope, or any part of either if a compound figure, which, under the general circumstances of the

conditions under which it arises, is a circle in its finite form, may, and often does, under particular circumstances of the conditions, assume the evanescent or infinite form of point or line (18): thus, for instance, the locus of a variable point or the envelope of a variable line whose distance from a fixed point is constant, which in general is the circle whose centre and radius are the point and constant, becomes of course evanescent or infinite when the constant $= 0$ or ∞.

2°. A locus or envelope, which, under the general circumstances of the conditions under which it arises, is a single figure of any form, often breaks up under particular circumstances of the conditions into two or more figures of simpler forms; thus, for instance, the locus of a variable point, the product of whose distances from any number of fixed lines, or the envelope of a variable line, the product of whose distances from any number of fixed points, is constant, which in general is a single figure of form depending on the number and disposition of the points or lines, breaks up into the entire system of lines or points when the constant $= 0$.

3°. A locus or envelope, which, under the general circumstances of the conditions under which it arises, is a definite determinate figure, simple or compound, becomes often *indeterminate* under particular circumstances of the conditions; thus, for instance, the locus of a variable point whose distances form two fixed lines, or the envelope of a variable line whose distances form two fixed points, are equal, which in general consists of the two lines or points of bisection of the angle or segment determined by the lines or points, becomes indeterminate when the lines or points coincide; *every* point in the former case, or line in the latter, then evidently satisfying the conditions of the locus or envelope.

As particular examples of loci and envelopes will appear in numbers in the course of the following pages, we shall not delay to give any here, but shall devote instead the remainder of the present chapter to the theory and properties of *similar figures* considered under their double aspect as loci of points and as envelopes of lines.

32. Two geometrical figures of any kind F and F', whether regarded as loci or envelopes, whose generating points or enveloping lines A, B, C, D, &c. and A', B', C', D', &c. correspond in pairs A to A', B to B', C to C', D to D', &c. are said to be *similar* when two points O and O', whether belonging to the figures or not, exist, such that for *every* two pairs of corresponding distances or perpendiculars OA and $O'A'$, OB and $O'B'$, the two angles AOB and $A'O'B'$ and the two ratios $OA : OB$ and $O'A' : O'B'$ are equal; and so also are two figures composed of systems of any common number of isolated points or lines, or mixed points and lines, A, B, C, D, &c. and A', B', C', D', &c. whose constituent elements correspond in pairs fulfilling the same conditions.

Two figures thus related to each other are said, like two hands or two feet, to be *both right or left* or *one right and the other left* according as the directions of rotation of the several pairs of corresponding angles AOB and $A'O'B'$, BOC and $B'O'C'$, COD and $C'O'D'$, &c. are similar or opposite.

As two angles, two ratios, or two magnitudes of any kind when equal to a third are equal to each other, it is evident from the conditions of similitude as above stated, that *two figures of any kind when similar to a third are similar to each other.*

33. Since, for two figures fulfilling the conditions of similarity, the ratios of the several pairs of corresponding distances or perpendiculars OA and $O'A'$, OB and $O'B'$, OC and $O'C'$, OD and $O'D'$, &c., by the second condition, are all equal, the constant value common to them all is termed the *ratio of similitude* of the figures; in the particular case when the ratio of similitude $=1$, that is, when the several pairs of corresponding distances or perpendiculars are all equal, the figures themselves also are said to be equal.

Since again, for two figures fulfilling the conditions of similarity, the angles between the several pairs of corresponding distances or perpendiculars OA and $O'A'$, OB and $O'B'$, OC and $O'C'$, OD and $O'D'$, &c., by the first condition, are all equal when the figures are both right or left, and all bisected by the same two rectangular directions when they are one right and the

other left, the constant value common to them all in the former case is termed the *angle of inclination*, and the fixed directions of bisection common to them all in the latter the *directions of symmetry* of the figures; when in the former case the angle of inclination $=0$ or $=$ two right angles, that is, when the directions of the several pairs of corresponding distances or perpendiculars (in both cases of course parallel) are all similar or opposite, the figures (in both cases said also to be parallel) are said to be *similarly* or *oppositely* placed.

34. From the preceding it is evident, conversely, that—

When two lines OA and $O'A'$, variable in length according to any law, turn in similar or opposite directions round two fixed extremities O and O', revolving simultaneously through equal angles and preserving as they revolve a constant ratio to each other, their two variable extremities A and A' describe, and the two perpendiculars to them at their variable extremities A and A' envelope, similar figures, whose ratio of similitude and angle of inclination or directions of symmetry are those of the lines.

For if A and A', B and B' be any two pairs of corresponding positions of the variable extremities, it follows at once from the conditions of revolution that the two angles AOB and $A'O'B'$ and the two ratios $OA : OB$ and $O'A' : O'B'$ are equal, and the two conditions of similarity of the figures described or enveloped being thus satisfied, the other circumstances respecting their ratio of similitude and angle of inclination or directions of symmetry are in fact stated in the conditions of revolution.

When the two fixed extremities O and O' coincide, and the two variable lines OA and $O'A'$ revolve in the same direction round the common extremity O, the species of the variable triangle AOA' is evidently constant, hence—

If one vertex of a triangle variable in magnitude and position but invariable in figure be fixed, the two variable vertices describe, and the two perpendiculars through them to the conterminous sides envelope similar figures, whose common ratio of similitude and angle of inclination are those of the variable sides containing the fixed vertex.

35. Two similar figures may be of such a form that a correspondence between their points or lines, in pairs satisfying the conditions of similarity, may exist *in more ways than one*, in the case of two *regular polygons* of any common order n, for instance, it may exist in n ways, and in the case of two *circles* in *an infinite number of ways*, and that whether the two figures be regarded as both right or left or one right and the other left. For, if O and O' be the centres of the two figures in either case, *any* pair or vertices or sides of the polygons, and *any* pair of points or tangents of the circles may be regarded as corresponding, and the correspondence between one pair of points or lines of the figures A and A' once established, that of all the remaining pairs B and B', C and C', D and D', &c. is of course fixed by the conditions that the several pairs of corresponding angles AOB and $A'O'B'$, AOC and $A'O'C'$, AOD and $A'O'D'$, &c., measured all either in similar or opposite directions of rotation, are equal. Such cases of similar figures are of course exceptional, but whenever they occur, as they necessarily do frequently in the geometry of the circle, their peculiarity in this respect leads sometimes to consequences not existing in the general case when the correspondence between the points or lines of the figures is unique.

36. In the particular cases when the radii of two circles regarded as similar figures are either evanescent or infinite; that is, when the two circles are either points or lines, *their ratio of similitude, being in all cases that of their radii, is indeterminate.* This peculiarity, which is evident on the general principles explained in (13), may easily be shewn, *a priori*, for both species of figures separately. For if I and I' be any two lines regarded as loci of points, or any two points regarded as envelopes of lines, O and O' in either case *any* two points taken arbitrarily, and A and A', B and B', C and C', &c. any number of pairs of points on the lines or of lines through the points, for which the several pairs of angles IOA and $I'O'A'$, IOB and $I'O'B'$, IOC and $I'O'C'$, &c. measured all in similar or opposite distances of rotation round O and O' are equal; since then in either case the several ratios $OA : O'A'$, $OB : O'B'$,

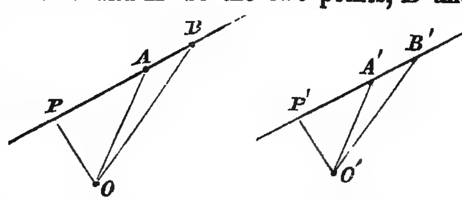
$OC : O'C'$, &c. are equal, the two figures are similar, and since in either case their common value $= OI : O'I'$, their ratio of similitude, the two points O and O' on which it depends being arbitrary, is indeterminate.

The preceding peculiarities of circles in general, and of points and lines in particular, regarded as similar figures, must always be carefully attended to in every application of the general theory of similar figures to their particular cases.

37. *For every pair of corresponding points of two similar figures F and F' regarded as loci, the two lines of connection with O and O' make equal angles and ratios with the two perpendiculars on their tangents from O and O'.*

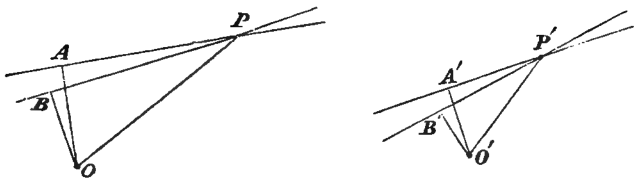
For every pair of corresponding tangents to two similar figures F and F' regarded as envelopes, the two perpendiculars from O and O' make equal angles and ratios with the two lines connecting their points of contact with O and O'.

To prove the first. If A and A' be the two points, B and B' any other pair of corresponding points, OP and $O'P'$ the two perpendiculars from O and O' upon the two indefinite lines



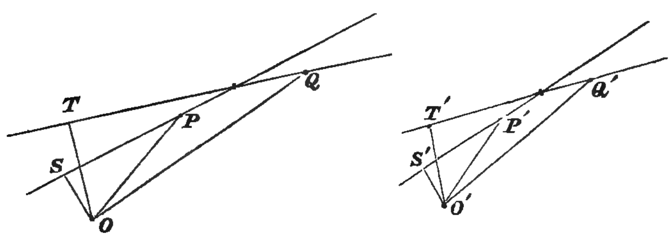
AB and $A'B'$, then since, whatever be the positions of the two pairs of corresponding points A and A', B and B', the two triangles AOB and $A'O'B'$ are by hypothesis similar (32), therefore the two triangles AOP and $A'O'P'$ are also similar, and therefore the two angles AOP and $A'O'P'$ and the two ratios $OA : OP$ and $O'A' : O'P'$ are equal; and this being true *in general*, whatever be the common magnitude of the two equiangular intervals AOB and $A'O'B'$, is *therefore* true *in the particular case* when that interval is evanescent, that is, when (19) the two lines AB and $A'B'$ are the two tangents to the two figures at the two points A and A'.

To prove the second. If A and A' be the two tangents, B and B' any other pair of corresponding tangents, OA and $O'A'$, OB and $O'B'$, the two pairs of perpendiculars upon them from O and O', and P and P' the two points of intersection of



A and B, and of A' and B'; then since, whatever be the positions of the two pairs of corresponding tangents A and A', B and B', the two triangles AOB and $A'O'B'$ are by hypothesis similar (32); therefore the two triangles AOP and $A'O'P'$ are also similar, and therefore the two angles AOP and $A'O'P'$, and the two ratios $OA : OP$ and $O'A' : O'P'$ are equal, and this being true *in general*, whatever be the common magnitude of the two equiangular intervals AOB and $A'O'B'$, is *therefore* true *in the particular case* when that interval is evanescent, that is when (20) the two points P and P' are the two points of contact with the two figures of the two tangents A and A'.

38. *When two figures regarded as loci of points are similar, they are also similar regarded as envelopes of lines, and conversely.*



For if P and P', Q and Q', be any two pairs of corresponding points, S and S', T and T', the two accompanying pairs of corresponding tangents; then since, by the preceding, the two pairs of angles POS and $P'O'S'$, QOT and $Q'O'T'$, and the two pairs of ratios $OP : OS$ and $O'P' : O'S'$, $OQ : OT$ and $O'Q' : O'T'$, are equal, when the figures whether regarded as loci or envelopes are similar; therefore the equality of the two angles POQ and $P'O'Q'$, and of the two ratios $OP : OQ$ and $O'P' : O'Q'$ involves that of the two angles SOT and $S'O'T'$, and of the two ratios $OS : OT$ and $O'S' : O'T'$, and conversely, and therefore &c.

39. When two figures F and F' are similar, every two points or lines X and X', whether belonging to the figures or not, which are such that for *any one* pair of points or lines of the figures A and A', the two angles AOX and $A'O'X'$ and the two ratios $OX : OA$ and $O'X' : O'A'$ are equal, are evidently, from the conditions of similarity (32), such that for *every other* pair B and B', the two angles BOX and $B'O'X'$ and the two ratios $OX : OB$ and $O'X' : O'B'$ are also equal. Every two such points or lines, whether belonging to the figures or not, are said to be *similarly situated*, and are termed *homologous points or lines*, with respect to the figures; all pairs, of corresponding points or lines A and A', of tangents T and T' at pairs of corresponding points P and P', and of points of contact P and P' of pairs of corresponding tangents T and T', of the figures, are evidently homologous.

From the nature of homologous points and lines as thus defined, it is evident for similar figures in general that—

1°. *If X and X' be any pair of homologous points or lines with respect to two similar figures F and F', the two distances or perpendiculars OX and OX' have the constant ratio of the similitude of the figures.*

For if A and A' be any pair of corresponding points or lines of the figures, since then, by hypothesis, $OX : OA = O'X' : O'A'$, therefore, by alternation, $OX : O'X' = OA : O'A'$, and therefore &c.

2°. *If X and X' be any pair of homologous points or lines with respect to two similar figures F and F', the two distances or perpendiculars OX and $O'X'$ have the same angle of inclination or directions of symmetry as the figures.*

For, if A and A' be any pair of corresponding points or lines of the figures, since then, by hypothesis, the two angles AOX and $A'O'X'$ are equal; therefore, according as their directions of rotation are similar or opposite, the two distances or perpendiculars OX and $O'X'$ have the same angle of inclination or directions of symmetry as the two OA and $O'A'$, and therefore &c.

3°. *If P and P', Q and Q' be any two pairs of homologous*

points with respect to two similar figures F and F′, the two connectors PQ and P′Q′ have the constant ratio of the similitude of the figures.

For, if A and A' be any pair of corresponding points or lines of the figures, since then, by hypothesis, the two pairs of angles AOP and $A'O'P'$, AOQ and $A'O'Q'$, and the two pairs of ratios $OP : OA$ and $O'P' : O'A'$, $OQ : OA$ and $O'Q' : O'A'$ are equal; therefore the two angles POQ and $P'O'Q'$ and the two ratios $OP : OQ$ and $O'P' : O'Q'$ are equal; and therefore, by similar triangles (Euc. VI. 4),

$$PQ : P'Q' = OP : O'P' = OQ : O'Q' = OA : O'A',$$

and therefore &c.

4°. *If P and P′, Q and Q′ be any two pairs of homologous points with respect to two similar figures F and F′, the two connectors PQ and P′Q′ have the same angle of inclination or directions of symmetry as the figures.*

For, if A and A' be any pair of corresponding points or lines of the figures, since then, by hypothesis, the two pairs of angles AOP and $A'O'P'$, AOQ and $A'O'Q'$, and the two pairs of ratios $OP : OA$ and $O'P' : O'A'$, $OQ : OA$ and $O'Q' : O'A'$ are equal; therefore the two angles of inclination of PQ to OA and of $P'Q'$ to $O'A'$ are equal; and, therefore, according as their directions of rotation are similar or opposite, the two connectors PQ and $P'Q'$ have the same angle of inclination or directions of symmetry as the two distances or perpendiculars OA and $O'A'$, and therefore &c.

5°. *If P and P′ be any pair of homologous points and L and L′ any pair of homologous lines with respect to two similar figures F and F′, the two perpendiculars PL and P′L′ have the ratio of similitude and the angle of inclination or directions of symmetry of the figures.*

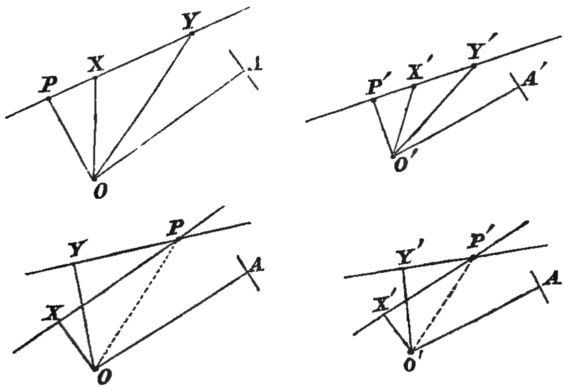
For, if A and A' be any pair of corresponding points or lines of the figures, since then, by hypothesis, the two pairs of angles AOP and $A'O'P'$, AOL and $A'O'L'$, and the two pairs of ratios $OP : OA$ and $O'P' : O'A'$, $OL : OA$ and $O'L' : O'A'$ are equal; therefore the two angles POL and

$P'O'L'$ and the two ratios $OP : OL$ and $O'P' : O'L'$ are equal; and therefore by pairs of similar right-angled triangles

$$PL : P'L' = OP : O'P' = OL : O'L' = OA : O'A',$$

and therefore &c; the second part being evident from the parallelism of PL and OL and of $P'L'$ and $O'L'$.

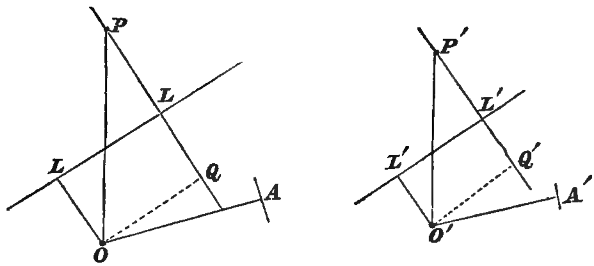
6°. *If X and X', Y and Y' be any two pairs of homologous points or lines with respect to two similar figures F and F', the two lines of connection or points of intersection XY and $X'Y'$ are homologous lines or points with respect to the figures.*



For, drawing the two perpendiculars or connectors OP and $O'P'$ from O and O' to XY and $X'Y'$. Since then for every pair of corresponding points or lines A and A' of the two figures F and F', the two pairs of angles AOX and $A'O'X'$, AOY and $A'O'Y'$, and the two pairs of ratios $OX : OA$ and $O'X' : O'A'$, $OY : OA$ and $O'Y' : O'A'$ are by hypothesis equal; therefore the two angles AOP and $A'O'P'$ and the two ratios $OP : OA$ and $O'P' : O'A'$ are equal, and therefore &c.

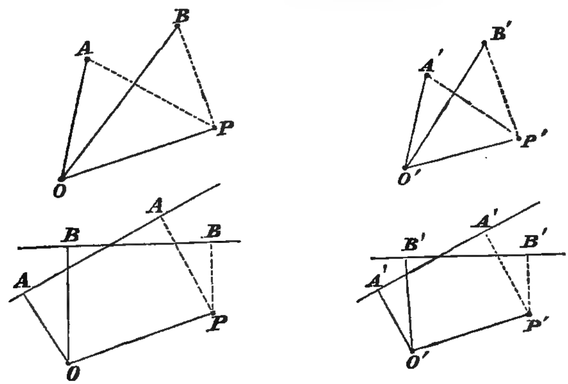
7°. *If P and P' be any pair of homologous points and L and L' any pair of homologous lines with respect to two similar figures F and F', the two perpendiculars PL and $P'L'$ are homologous lines and their two intersections with L and L' are homologous points with respect to the figures.*

For, drawing from O and O' the two perpendiculars OQ and $O'Q'$ to PL and $P'L'$. Since then for every pair of



corresponding points or lines A and A' of the two figures F and F', the two pairs of angles AOP and $A'O'P'$, AOL and $A'O'L'$, and the two pairs of ratios $OP : OA$ and $O'P' : O'A'$, $OL : OA$ and $O'L' : O'A'$, are by hypothesis equal, therefore the two angles AOQ and $A'O'Q'$ and the two ratios $OQ : OA$ and $O'Q' : O'A'$ are equal, and therefore &c.; the second part following from the first by the second part of 6°.

8°. *Any two homologous points P and P' with respect to two similar figures F and F' may be substituted for the two O and O' without violating the conditions of similitude of the figures.*



For, if A and A', B and B' be any two pairs of corresponding points or lines of the figures; then since by hypothesis the two pairs of angles AOP and $A'O'P'$, BOP and $B'O'P'$, and the two pairs of ratios $OA : OP$ and $O'A' : O'P'$, $OB : OP$ and $O'B' : O'P'$ are equal; therefore, by pairs of similar triangles, the two angles APB and $A'P'B'$ and the two ratios $PA : PB$ and $P'A' : P'B'$ are equal, and therefore &c., (32).

9°. *For every two similar figures F and F' if any number of points connected with either F lie on a line L, the homologous points with respect to the other F' lie on the homologous line L', and, if any number of lines connected with either F pass through a point P, the homologous lines with respect to the other F' pass through the homologous point P'.*

For, since by 5°, for every pair of homologous points P and P', and for every pair of homologous lines L and L', of the figures, $PL : P'L' =$ the constant ratio of similitude of F and F', therefore if either of them $= 0$ so also is the other, that is, if the point P lie on the line L the point P' lies on the line L', and if the line L pass through the point P the line L' passes through the point P', and therefore &c.

10°. *For every two similar figures F and F', if any number of points or lines connected with either F lie on or touch a circle C, the homologous points or lines with respect to the other F' lie on or touch a circle C', the centres of the two circles being homologous points and their radii having the ratio of similitude of the figures.*

For, since by 3° and 5° or by 8°, for every pair of homologous points P and P', and for any number of pairs of homologous points or lines X and X', Y and Y', Z and Z', &c. of the figures $PX : P'X' = PY : P'Y' = PZ : P'Z'$, &c. $=$ the constant ratio of similitude of F to F', therefore if $PX = PY = PZ$, &c., that is if X, Y, Z, &c. lie on or touch a circle of which P is the centre and their common distance from it the radius, then $P'X' = P'Y' = P'Z'$, &c., that is X', Y', Z', &c. lie on or touch a circle of which P' is the centre and their common distance from it the radius, and therefore &c.

11°. *If a pair of homologous points or lines X and X' with respect to two similar figures F and F' vary simultaneously according to any law, the two figures G and G' they describe or envelope are similar and have the same ratio of similitude and the same angle of inclination or directions of symmetry as the original figures.*

For, if A and A' be any pair of corresponding points or lines of F and F', then since in every position of the two variable homologues X and X', the two angles AOX and

$A'O'X'$ and the two ratios $OX : O'X'$ and $OA : O'A'$ are equal, therefore &c., (32). This general property, here established on general principles, includes of course the particular cases 9° and 10° established above by particular considerations.

Every two figures G and G' described or enveloped as above are said to be *homologous figures* with respect to the originals F and F', which again reciprocally are evidently homologous figures with respect to G and G', and every pair of points P and P', of lines L and L', of circles C and C', and generally of figures of any kind E and E', which are homologous with respect to either pair F and F' are evidently also homologous with respect to the other pair G and G', and conversely.

40. *If a figure of any invariable form revolve round any point invariably connected with it as a fixed centre, varying in magnitude as it revolves according to any law, all points invariably connected with it describe, and all lines invariably connected with it envelope, similar figures, all right or left, whose ratios of similitude and angles of inclination two and two are those of the distances of the describing points or enveloping lines from the fixed centre.*

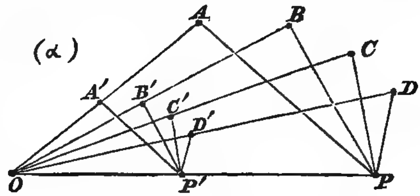
For, if O be the fixed point, and X, Y, Z, &c. any number of variable points or lines all invariably connected with the variable figure; then since the form of the figure, whatever be the law of its variation in magnitude while revolving round O, is by hypothesis invariable, therefore, by the preceding (39), the several angles XOY, YOZ, &c., and the several ratios $OX : OY$, $OY : OZ$, &c. are all constant, and therefore &c., (32).

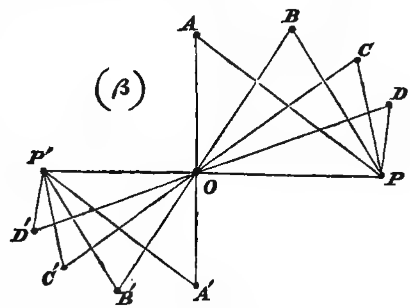
For points and lines of the revolving figure not evanescently or infinitely distant from O, it is easy to verify by particular considerations as in 9° and 10° of the preceding article, that in particular, *if any one point P describe a line or circle all points P, Q, R, &c. describe lines or circles, and if any one line L envelope a point or circle all lines L, M, N, &c. envelope points or circles;* this verification, there gone through in detail, need not of course be repeated here.

41. *When two similar figures of any kind, both right or left, are similarly or oppositely placed* (33), *all lines AA', BB',*

CC', DD', &c. connecting pairs of corresponding points pass through a common point O, and are there cut, externally or internally, in the ratio of the similitude of the figures.

For if O be the point in which any one of them AA' intersects the line PP', connecting any pair of homologous points P and P' with respect to the figures; since then, by hypothesis (33), the two directions PA and $P'A'$, whether similar (fig. α) or opposite (fig. β), are parallel; therefore, by similar triangles, the two ratios $OP : OP'$ and $OA : OA'$ are each = the ratio $PA : P'A'$ = the ratio of similitude of the figures; therefore all connectors AA', BB', CC', DD', &c. cut and are cut by the same line PP' at the same point O, and in the same ratio $OP : OP'$, and therefore &c.





Conversely, *if the several lines connecting any arbitrary point O with all the points A, B, C, D, &c. of a figure of any kind, be increased or diminished in similar or opposite directions in any common ratio, the several extremities A', B', C', D', &c. of the increased or diminished distances determine a second figure similar to the original, and similarly or oppositely placed with it according as the directions of the original and altered distances are similar or opposite.*

For, every pair of corresponding angles AOB and $A'OB'$ and every pair of corresponding ratios $OA : OB$ and $OA' : OB'$ being equal, the figures are similar; and every pair of corresponding directions OA and OA', OB and OB', OC and OC', &c. being similar or opposite, the figures are similarly or oppositely placed (33), and therefore &c.

Since, by pairs of similar triangles AOB and $A'OB'$, the two lines AB and $A'B'$ connecting any two points A and B

of either figure, and the two corresponding points A' and B' of the other are always parallel, whatever be the angle between the two lines AA' and BB' passing through O, they are therefore so in the particular case where that angle $=0$, that is, when AA' and BB' coincide and when therefore (19) AB and $A'B'$ are the two tangents to the figures at A and A'. Hence, *when two similar right or left figures are similarly or oppositely placed, all pairs of tangents at pairs of corresponding points, like all other pairs of homologous lines of the figures, are parallel.*

42. The point O related as above to two similar right or left figures, when similarly or oppositely placed, is termed their *centre of similitude*, and is said to be *external* or *internal*, with respect to them, according as the section by it of all lines connecting pairs of homologous points in the common ratio of their similitude is external or internal, that is, according as they are similarly or oppositely placed; when the two figures in either case are given in absolute position, their centre of similitude O is evidently given by the intersection of any two lines PP' and QQ' connecting pairs of homologous points on or in any way situated with respect to them.

As all lines connecting pairs of homologous points P and P', Q and Q', R and R', S and S', &c., situated in any manner with respect to the figures, pass through O, and are there cut in the ratio of their similitude, externally or internally, according as their positions are similar or opposite; so, conversely, all pairs of points P and P', Q and Q', R and R', S and S', &c., which connect by lines passing through O, and there cut in their ratio of similitude, externally or internally according as their positions are similar or opposite, are evidently homologous pairs with respect to the figures; and the two similar and similarly or oppositely placed figures $PQRS$ &c. and $P'Q'R'S'$ &c., determined by any number of such pairs, are evidently similarly situated with respect to, and have the same centre and ratio of similitude with, the original figures $ABCD$ &c. and $A'B'C'D'$ &c.

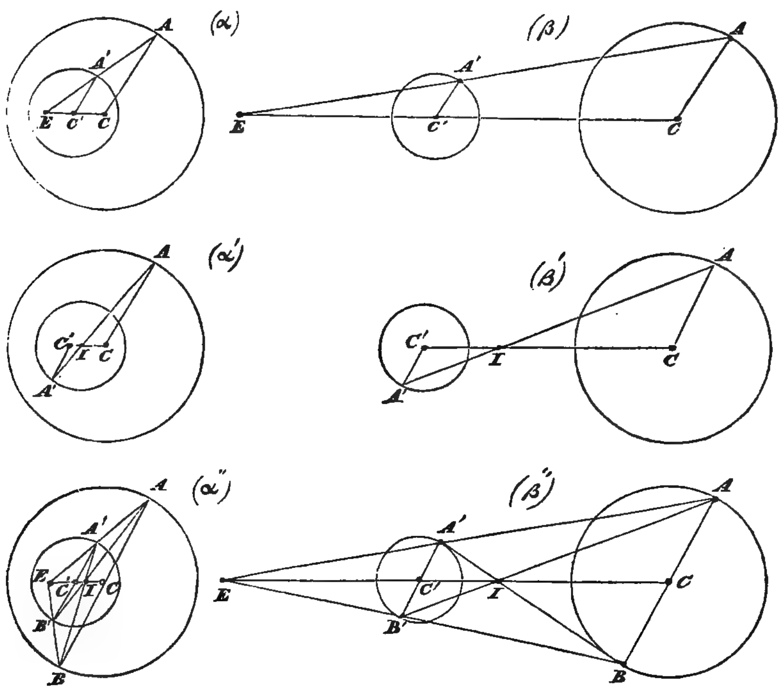
Every line passing through O being evidently its own homologue with respect to both figures and intersecting them,

if it meet them at all, at pairs of corresponding points A and A', B and B', C and C', &c., of which the number depends, of course, on the nature of the figures; at which the several pairs of corresponding tangents, by (41), are parallel; and for which the several pairs of ratios $OA : OA'$, $OB : OB'$, $OC : OC'$, &c., by the same, are equal to the ratio of similitude of the figures, so that if $OA = OB$, or any two of the points of meeting for either figure coincide, then also $OA' = OB'$, or the two corresponding points of meeting for the other also coincide. Hence, *when two similar right or left figures are similarly or oppositely placed, every line passing through their centre of similitude, like every pair of homologous lines in general, divides and is divided by them similarly into pairs of corresponding segments in the linear ratio of their similitude, intersects them at equal angles at every pair of corresponding points of meeting, and if it touch either at any point of meeting touches the other also at the corresponding point of meeting.*

43. Two similar figures of such a form, that a correspondence between their points and lines in pairs satisfying the conditions of similarity, exists in more ways than one (35), may be, moreover, of such a form that when similarly placed for one mode of correspondence, they are at the same time oppositely placed for another, or conversely; as for instance, *two similar parallelograms*, or, more generally, *two similar polygons of any even degree whose several pairs of opposite sides are equal and parallel*; every two such figures when thus at once similarly and oppositely placed have of course *two different centres of similitude*, one external corresponding to their similar, and the other internal corresponding to their opposite, parallelism, each determined, as in the general case, by the intersection of any two lines connecting pairs of homologous points for the relative positions corresponding to itself, and each possessing all the properties of the unique centre of similitude of the same kind with itself in the general case.

44. Of figures coming under the above head *two circles*, however circumstanced as to magnitude or position, absolute

or relative, provided only they be in the same plane, possess, for the reason explained in (35), the property, confined to them exclusively, of being *always* at once similarly and oppositely placed, and of having therefore in *every* position two



different centres of similitude, one E external as similarly placed, figs. (α) and (β), and the other I internal as oppositely placed, figs. (α') and (β'); both situated on the line CC' connecting their centres C and C' and dividing that line, the former externally and the latter internally, in the ratio of their radii; both determined by the intersections with that line of the lines AA' connecting the extremities of any two parallel radii CA and $C'A'$ drawn in similar directions, figs. (α) and (β), for the former, and in opposite directions, figs. (α') and (β'), for the latter—or, which comes to the same thing, by the intersections with each other of the pairs of lines AA' and BB', AB' and BA', connecting the extremities,

adjacent for the former and non-adjacent for the latter, of any two parallel diameters AB and $A'B'$, figs. (α'') and (β''); and each possessing, as for every two figures coming under the same head, all the properties of the unique centre of similitude of its kind for any two similar figures similarly or oppositely placed, (41) and (42).

As every line touching two circles in the same plane connects the extremities of the two parallel radii to which it is perpendicular (Euc. III. 18); and consequently, by the above, passes through either the external or the internal centre of similitude of the circles, according as the directions of the radii are similar or opposite; hence two circles in the same plane, however circumstanced as to magnitude and position, admit, in general, of two, and of but two, pairs of common tangents, real or imaginary, both symmetrically situated with respect to, and intersecting upon, their line of centres; one, termed in consequence the external pair, intersecting at their external centre of similitude, and the other, termed in consequence the internal pair, intersecting at their internal centre of similitude; and, evidently, both real, both imaginary, or, one real and one imaginary, according as the distance between their centres is greater than the sum, less than the difference, or, intermediate between the sum and difference, of their radii.

The two centres of similitude, external and internal, of two given circles, determined as above, or by any other method, give, consequently, in two conjugate pairs (Euc. III. 17), the four solutions, real or imaginary, of the problem "To draw a common tangent to the two circles."

CHAPTER III.

THEORY OF MAXIMA AND MINIMA.

45. When a geometrical magnitude of any kind, which varies continuously according to any law, passes in the course of its variation through a value greater than either its preceding or succeeding values, it is said to be *a maximum*, even though at some other stage of its variation it may pass through a value absolutely greater; and, on the other hand, when it passes in the course of its variation through a value less than either its preceding or succeeding values it is said to be *a minimum*, even though at some other stage of its variation it may pass through a value absolutely less; the terms "maximum" and "minimum," as employed in geometry, are therefore relative, not absolute.

46. As, to a traveller on a road which is not a dead level, the top of *every* hill is *a* position of maximum, and the bottom of *every* hollow *a* position of minimum, elevation above the sea or any other standard level; so, for geometrical figures of the higher orders, the different variable magnitudes connected with them, may pass in the course of their variation through several maxima and several minima values, of course necessarily alternating with each other in the order of their occurrence; as, for instance, the linear distance from any fixed point, or the perpendicular distance from any fixed line, of a variable point, traversing the entire figure or any part of it; for the point, line, and circle, however the variable magnitudes most commonly considered in connection with them and their combinations, rarely pass during their variations through more than a single maximum and a single minimum value; as, for instance, the distance of a variable point on

a circle from any fixed point or line situated in any manner with respect to it, which, in either case (Euc. III. 7, 8, 19), is a maximum for and only for the distance which passes through the centre, and a minimum for and only for the distance which if produced would pass through it; in all such cases the single maxima and minima values are not only relatively but also absolutely the greatest and least values through which the variable magnitude passes in the course of its variation.

47. As every increase or diminution of a magnitude of any kind is necessarily accompanied by the simultaneous diminution or increase of its reciprocal (8); it follows, of course, that when a variable magnitude passes under any circumstances through a maximum or minimum value, its reciprocal to any unit, passes simultaneously through a minimum or maximum value.

48. The following are a few simple but fundamental examples of maxima and minima, to which many others are reducible:—

Ex. 1°. *When two sides of a triangle are given in magnitude the area is a maximum (in this case the maximum) when they contain a right angle.*

For (Euc. I. 41), whatever be their angle of intersection, acute, right, or obtuse, the area = half the product of either into the perpendicular on its direction from the remote extremity of the other, which perpendicular is evidently equal to the other for the right and less than the other for any position at either side of the right angle; and, in the same way generally, when one side of a triangle is constant the area varies as, and therefore passes through, its maxima and minima values with the perpendicular upon its direction from the opposite vertex.

Ex. 2°. *For the point of internal bisection of any segment of a line the product of the distances from the extremities is a maximum, and the sum of their squares a minimum.*

For (Euc. II. 5 and 9, 10), the product for that point exceeds the product for any other point of internal section on either side by the square, and the sum of the squares for that point falls short of the sum of the squares for any other point of section, external or internal, on either side, by twice the square, of the distance between that and the other point of section; and, in the same way generally, for any two magnitudes expressed in numbers, as product = square of half sum − square of half difference, and as sum of squares = twice square of half sum + twice square of half difference; if the sum be constant, the product

is a maximum and the sum of the squares a minimum; and if the product or the sum of the squares be constant, the sum is a minimum in the former case and a maximum in the latter, when the magnitudes are equal.

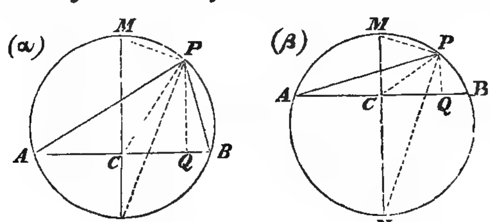
Ex. 3°. *For any two magnitudes expressed in numbers whose sum is constant, the sum, product, sum of squares, and product of squares, of the reciprocals are all minima when the magnitudes are equal.*

For, the product of the reciprocals being = the reciprocal of the product, and the product of the squares of the reciprocals being = the reciprocal of the square of the product are both minima when the product is a maximum, that is, Ex. 2°, when the magnitudes are equal; and again, the sum of the reciprocals being = the sum divided by the product, and the sum of the squares of the reciprocals being = the sum of the squares divided by the product of the squares, are both minima also, when the product is a maximum, the sum being constant by hypothesis, and the sum of the squares being then a minimum, Ex. 2°.

Ex. 4°. *For the point of internal bisection of any side of a triangle the area of the inscribed parallelogram formed by drawing parallels to the other two sides is a maximum.*

For, whatever be the position of the point of section, the angle of the parallelogram being constant, its area (Euc. VI. 23) varies as the product of the parallels; that is, as the product of the segments of the divided side determined by the point of section, the former being to the latter product in the constant ratio of the rectangle under the other two sides to the square of that side (Euc. VI. 23); but the latter product being a maximum, by Ex. 2°, for the point of bisection of the side, so therefore is the former, and therefore the area of the parallelogram; and, in the same manner exactly, it appears that, for the point of internal bisection of any side of the triangle the product of the perpendiculars on the other two sides, or more generally of the two lines drawn in any two given directions to meet them, is a maximum.

Ex. 5°. *For the point of internal bisection of any arc of a circle, the sum of the squares of the linear distances from the extremities is a maximum or a minimum, and for the point of external bisection a minimum or a maximum, according as the arc is greater or less than a semicircle.*



For, if AB be the arc, C the middle point of its chord, M and N its two points of bisection, internal and external, P any other point on the

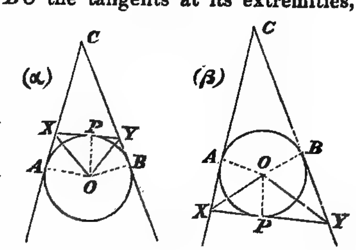
circle, and PQ the perpendicular from P on AB; then since (Euc. II. 12, 13), whatever be the position of P, $PA^2 = PC^2 + CA^2 \pm 2CA.CQ$, and $PB^2 = PC^2 + CB^2 \mp 2CB.CQ$, therefore $PA^2 + PB^2 = CA^2 + CB^2 + 2.CP^2$, which is a maximum or a minimum when CP is a maximum or a minimum; that is (Euc. III. 7), when P is at M or N in the former case, and at N or M in the latter; and, in the same manner, it appears generally that the sum of the squares of the linear distances of a variable point P, on any geometrical figure from any two fixed points A and B, situated in any manner with respect to the figure, increases and diminishes and passes through its maxima and minima values, with the distance PC of the variable point P from the middle point C of the line AB connecting the two fixed points A and B.

Ex. 6°. *For each point of bisection, internal and external, of any arc of a circle, the sum and product of the linear distances from the extremities, and the area of the triangle they determine with the chord, are all maxima.*

For, since whatever be the position of P, (same figures as in last), $PA.PB = MN.PQ$ (Euc. VI. 16), and area $APB = \frac{1}{2}AB.PQ$ (Euc. I. 41); the property is evident as regards the product and area, and it remains only to prove it for the sum $PA + PB$, which is easily done as follows: since for every position of P at the same side of the chord with M (as in the figures), by Ptolemy's Theorem (Euc. VI. 16, Cor.), $PA.NB + PB.NA = PN.AB$, and since, by hypothesis, $NA = NB$, therefore $PA + PB : PN :: AB : AN$ or BN, that is, in a constant ratio, and therefore $PA + PB$ is a maximum when PN is a maximum, that is, when P is at M; and in the same way it may be shewn (by simply substituting M for N in the above) that for positions of P at the same side of AB with N, $PA + PB$ varies as PM, and is therefore a maximum when P is at N.

Ex. 7°. *For each point of bisection, internal and external, of any arc of a circle, the segment of the tangent intercepted between the tangents at the extremities, and the area of the triangle it subtends at the centre of the circle, are both minima.*

For, if AB be the arc, AC and BC the tangents at its extremities, XY the segment intercepted between them of the tangent at any other point P, and O the centre of the circle; then, since whatever be the position of P, the lines OX and OY bisect the angles AOP and BOP (Euc. III. 17), the angle between them, XOY is equal to half the angle AOB subtended at O by the arc APB, therefore



in the triangle whose vertex is O and base XY, the altitude OP and vertical angle XOY are both constant; and it is evident from the preceding, or independently, that when the vertical angle of a triangle is constant, the

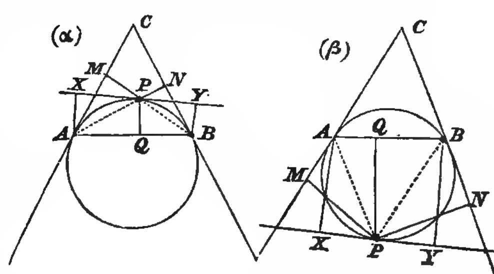
altitude and area are both maxima for a given base, and the base and area both minima for a given altitude, when the triangle is isosceles, that is, for the triangle XOY when P is a point of bisection, internal or external, of the arc AB.

Ex. 8°. *For the point of internal bisection of any arc of a circle, the area of the triangle formed by the tangent with the tangents at the extremities is a maximum or a minimum, and for the point of external bisection a minimum or a maximum, according as the arc is less or greater than a semicircle.*

For, since in either case (same figures as in last), the pentagonal area $XAOBY$, being double the triangular area XOY, is a minimum, by the preceding, for each point of bisection of AB; and the quadrilateral area $AOBC$ being of course constant, whatever be the position of XY, therefore the tri ngular area XCY, being = the quadrilateral − the pentagon in one case (fig. α), and = the quadrilateral + the pentagon in the other case (fig. β), is a maximum in the former case and a minimum in the latter.

Ex. 9°. *For each point of bisection, internal and external, of any arc of a circle, the product of the perpendiculars upon the tangents at the extremities, and the product of the perpendiculars from the extremities upon the tangent, are both maxima.*

For, if AB be the arc, P any point upon it, external or internal, PM and PN the perpendiculars from P upon the tangents at A and B, AX and BY the perpendiculars from A and B upon the tangent at P, and PQ the perpendicular from P upon the chord AB; then, joining P with A and B, by pairs of equal triangles

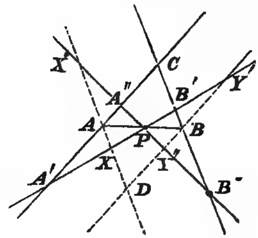


APM and PAX, BPN and PBY, we have $PM = AX$ and $PN = BY$, and therefore $PM.PN = AX.BY$, and by pairs of similar triangles APM (or PAX) and BPQ, BPN (or PBY) and APQ (Euc. III. 32), we have PM or $AX : PQ :: PQ : PN$ or BY, both being $= PA : PB$, therefore $PM.PN$ and $AX.BY$ both $= PQ^2$, and therefore &c.

Ex. 10°. *Of all lines passing through a fixed point that which determines with two fixed lines the triangle of minimum area is that whose segment intercepted between the lines is bisected at the point.*

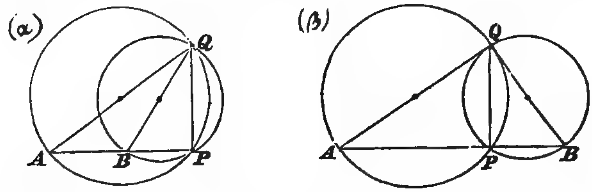
For, if P be the point, AC and BC the lines, AB the intercept bisected at P, and $A'B'$ or $A''B''$ any other intercept; then through A and B drawing AD and BD parallels to BC and AC, meeting $A'B'$

or $A''B''$ at X' and Y' or X'' and Y''. As the two triangles APX' and BPB', or the two triangles BPY'' and APA'', are evidently equal (Euc. I. 4); therefore the triangle ACB is less than the triangle $A'CB'$ or $A''CB''$, and therefore &c.



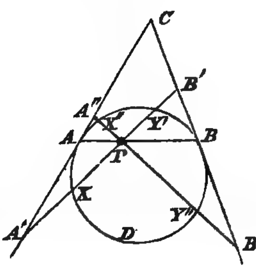
The point and lines being given, to draw AB so as to be bisected at P, is, of course, but a particular case of the more general problem to draw it so as to be cut in any given ratio, of which the preceding construction suggests the following obvious solution: drawing from P any line PA' or PA'' to either line CA, and producing it through P to Y' or Y'' so that $PA' : PY'$ or $PA'' : PY''$ = the given ratio, the parallel $Y'B$ or $Y''B$ to CA through Y' or Y'' evidently intersects the other line CB in the extremity B of the required line AB.

Ex. 11°. *Of all lines passing through either point of intersection of two circles, that whose segment intercepted between the circles is of maximum length, and subtends at the other point of intersection, the triangle of maximum area is that which is perpendicular to the chord of intersection.*



For, if PQA and PQB be the circles, P and Q their points of intersection, and AB any line passing through either of them P and meeting the circles at A and B; then since, joining A and B with the other intersection Q, the angles PAQ and PBQ are both constant (Euc. III. 21), the triangle AQB is constant in species, whatever be the position of AB, and therefore its base AB, area AQB, and sides QA and QB are all maxima together; but the sides QA and QB are maxima when they are diameters of their respective circles, that is (Euc. III. 31) when AB is perpendicular to PQ.

Ex. 12°. *Of all lines passing through a fixed point that whose segments intercepted in opposite directions between the point and two fixed lines contain the rectangle of minimum area is that which makes equal angles with the lines.*

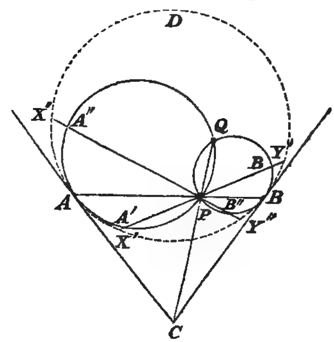


For, if P be the point, AC and BC the lines, AB the line through P making equal angles with AC and BC, and $A'B'$ or $A''B''$ any other line through P; then, as evidently

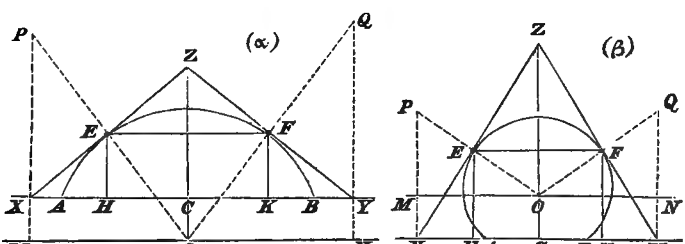
the circle ADB touching AC and BC at A and B intersects $A'B'$ or $A''B''$ at points X' and Y' or X'' and Y'' internal to A' and B' or A'' and B'', the rectangle $PA.PB$ which is equal to the rectangle $PX'.PY'$ or $PX''.PY''$ (Euc. III. 35), is therefore less than the rectangle $PA'.PB'$ or $PA''.PB''$, and therefore &c.

Ex. 13°. *Of all lines passing through either point of intersection of two circles that whose segments intercepted in opposite directions between the point and circles contain the rectangle of maximum area is that which makes equal angles with the circles* (22).

For, if PAQ and PBQ be the circles, P and Q their two points of intersection, AB the line passing through either of them P making equal angles with the circles, that is (22) with the tangents to them AC and BC at its extremities A and B, and $A'B'$ or $A''B''$ any other line through P; then, as evidently the circle ADB touching AC and BC at A and B intersects $A'B'$ or $A''B''$ at points X' and Y' or X'' and Y'' external to A' and B' or A'' and B''; the rectangle $PA.PB$ which is equal to the rectangle $PX'.PY'$ or $PX''.PY''$ (Euc. III. 35) is therefore greater than the rectangle $PA'.PB'$ or $PA''.PB''$, and therefore &c.



Ex. 14°. *The rectangle of maximum area inscribed in any segment of a circle, or of any other convex figure, is that whose side parallel to the base of the segment bisects the sides of the triangle formed with the base by the lines touching at its extremities the circle or figure.*



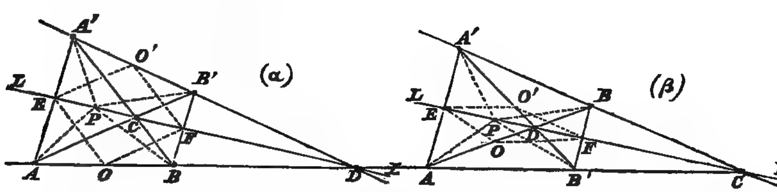
For, if $AEFB$ be the segment, EF the chord parallel to its base AB, which bisects the sides XZ and YZ of the triangle XZY formed with AB by the tangents at E and F; then, by Ex. 4°, the rectangle (or parallelogram) $EFKH$ is the maximum that could be inscribed in the triangle XZY, and therefore, *a fortiori*, in the segment $AEFB$ to which the triangle is external.

To draw EF so as to bisect the tangents ZX and ZY is, of course, a particular case of the more general problem, to draw it so as to cut them in any given ratio, which for the circle may be done as follows: through the centre O drawing OC and MN perpendicular and parallel to AB (the former of course passing through Z), and through X and Y, supposed found, drawing XM and YN parallel to OC to meet the radii OE and OF, supposed found, at P and Q respectively; then by pairs of similar triangles PEX and OEZ, QFY and OFZ, the two ratios $PE : EO$ and $QF : FO$ each = the given ratio of the tangents, and therefore as EO and FO are given and equal, PE and QF, PO and QO, and the rectangles $PE.PO$ and $QF.QO$, are given and equal; but by other pairs of similar triangles PEX and PMO, QFY and QNO, $PM.PX = PE.PO$, and $QN.QY = QF.QO$, therefore the rectangles $PM.PX$ and $QN.QY$ are given and equal; but MX and NY, being each = CO, are also given and equal; therefore (Euc. II. 6) PM and QN, PX and QY, and the angles POM and QON are given and equal, and therefore E and F are known.

49. The next example we give separately as the basis of some useful properties of the triangle.

a. The lines connecting a variable point on a fixed line with two fixed points at the same side of the line have the maximum difference when they coincide in direction, and the minimum sum when the angle between them is bisected (of course externally) by the line.

b. The lines connecting a variable point on a fixed line with two fixed points at opposite sides of the line have the minimum sum when they coincide in direction, and the maximum difference when the angle between them is bisected (of course internally) by the line.



Let LL, figs. α and β, be the fixed line, A and B the two fixed points, AE and BF the two perpendiculars from them on LL, A' and B' the two points on the perpendiculars for which $AE = EA'$ and $BF = FB'$, then the distances of any point P on LL from A and A', or from B and B', being equal

(Euc. I. 4), if D be the point on it at which AB or $A'B'$ intersects it, that is the point on it for which PA and PB coincide in direction, and if C be the point on it at which AB' or $A'B$ intersects it, that is the point on it for which the angle APB is bisected (externally fig. α, or internally fig. β) by it; it is to be shewn that, in fig. α, $DA \sim DB > PA \sim PB$, and $CA + CB < PA + PB$, and that, in fig. β, $DA + DB < PA + PB$, and $CA \sim CB > PA \sim PB$, which are evident, the first for each figure from the triangle APB or $A'PB'$, and the second for each figure from the triangle APB' or $A'PB$, any side of a triangle (Euc. I. 20) being greater than the difference and less than the sum of the other two.

The maximum difference in a (fig. α), or minimum sum in b (fig. β), is of course the distance AB between the two points A and B; the minimum sum in a (fig. α), or maximum difference in b (fig. β), may be expressed in terms of the distances of the points from the line and from each other as follows:

In both cases the four points $ABA'B'$ lie evidently in a circle, and the two pairs of opposite connectors AB and $A'B'$, AB' and $A'B$ are evidently equal; therefore, by Ptolemy's Theorem (Euc. VI. 16, Cor.), $AA'.BB' = AB'.A'B - AB.A'B'$ in fig. α, and $= AB.A'B' - AB'.A'B$ in fig. β; but $AA' = 2.AE$, $BB' = 2.BF$, and $AB' = A'B = AC + BC$ in fig. α, and $= AC \sim BC$ in fig. β; therefore

$$(AC + BC)^2 = AB^2 + 4.AE.BF\ldots\ldots\text{in fig. } \alpha,$$

and

$$(AC \sim BC)^2 = AB^2 - 4.AE.BF\ldots\ldots\text{in fig. } \beta,$$

which are the formulæ by which to calculate in numbers the minimum sum or maximum difference when the distances of the points from the line and from each other are given.

The line LL being in fig. α the external and in fig. β the internal bisector of the vertical angle C of the triangle ACB, we see from the above formulæ that—

If from the extremities of the base of a triangle perpendiculars be let fall upon the external or internal bisector of the vertical angle, their rectangle = square of half sum of sides − square of half base in the former case, and = square of half base − square of half difference of sides in the latter case.

If the interval AB between the two points A and B be

bisected or conceived to be bisected at O, and the point of bisection O connected or conceived to be connected with the feet of the two perpendiculars E and F; then, evidently, $OE = \frac{1}{2}BA'$ and $OF = \frac{1}{2}AB'$, therefore $OE = OF = \frac{1}{2}(AC + BC)$ in fig. α, and $= \frac{1}{2}(AC \sim BC)$ in fig. β. Hence—

If from the extremities of the base of a triangle perpendiculars be let fall upon the external or internal bisector of the vertical angle, their feet are equidistant from the middle point of the base by an interval = half the sum of the sides in the former case, and = half the difference of the sides in the latter case.

From these last two properties combined we see that, when the base of a triangle is fixed and the sum or difference of the sides constant, if perpendiculars be let fall from the extremities of the base upon the external or internal bisector of the vertical angle—

a. Their feet are equidistant from the middle point of the base by a constant interval = half the sum or difference of the sides.

b. Their rectangle is constant and = square of half sum or difference of sides ~ square of half base.

The interval EF between the feet of the perpendiculars being a chord of the circle round O as centre, whose radius $= \frac{1}{2}(AC + BC)$ in fig. α, and $= \frac{1}{2}(AC \sim BC)$ in fig. β, and the square of the semi-interval AB between the two points A and B being = the square of the radius of the circle $\mp$ the rectangle $AE.BF$, we see that—

The two perpendiculars erected at the extremities of any chord of a circle meet any diameter of the circle at two points equidistant from the centre and contain a rectangle = the square of the radius of the circle ~ the square of the semi-interval they intercept on the diameter.

A useful property of the circle which the reader may very easily prove, *à priori*, for himself.

50. If from any point A a perpendicular be let fall upon any line L, and produced, as in the preceding, through the line to a second point A' equidistant from L with A, the new point A' is termed *the reflexion* of the original point A with respect to the line L; and, generally, if from all the points A, B, C, D, &c. of any geometrical figure perpendiculars be let

fall upon any line L and produced through L, in the same manner, to their reflexions at the opposite side, the new figure A', B', C', D', &c. is termed *the reflexion* of the original with respect to the line.—A convenient term introduced into Geometry from the science of Optics.

The relation between any figure and its reflexion with respect to any line is evidently *reciprocal* (8); that is, if one figure F' be the reflexion of another F with respect to a line L, the latter F is reciprocally the reflexion of the former F' with respect to the same line L; it is evident also, that every two figures F and F' reflexions of each other with respect to any line L are *right and left figures* (32), *similar* in *form*, *equal* in *magnitude*, and *symmetrically situated*, like two hands or two feet, with respect to the line and to each other.

Thus, the reflexion of a line is a line, of a circle a circle, of the line passing through two points the line passing through the reflexions of the points, of the circle passing through three points or touching three lines the circle passing through the reflexions of the points or touching the reflexions of the lines, &c.; and, generally, of any figure intersecting or touching another a similar and equal figure touching or intersecting the reflexion of the other at the reflexions of the point or points of intersection or contact of the original figures.

All points common to two figures reflexions of each other lie of course on the line (or *axis* as it is sometimes termed) of reflexion, which evidently bisects at once all the angles finite or evanescent at which they intersect or touch each other.

Every circle having its centre on the axis of reflexion of any two figures reflexions of each other evidently intersects or touches both, when it meets them at all, at pairs of points reflexions of each other with respect to the axis; this peculiarity of the circle arises from the evident circumstance that every diameter of the figure divides it into two halves reflexions of each other with respect to itself.

If the plane of any figure be turned round any line in itself through an angle of 180°, the figure in the new is evidently the reflexion of itself in the old position with respect to the line.

From properties a and b of the preceding article it appears that—

When the lines connecting a variable point on a fixed line with two fixed points not on the line are reflexions of each other with respect to the line, their sum is a minimum or their difference a maximum according as the points lie at the same or at opposite sides of the line.

51. The next example again we give separately as the basis of some important properties of the circle.

The lines connecting a variable point on a fixed line, circle, or any other geometrical figure with two fixed points situated in any manner with respect to the figure, contain a maximum or minimum angle for every point at which a variable circle passing through the points touches the figure.

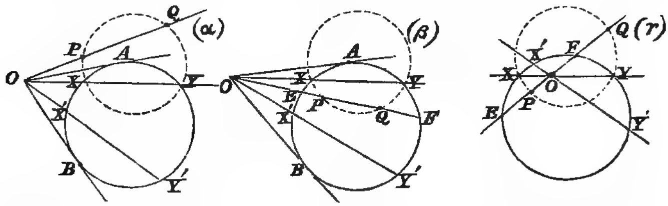
For, (Euc. III. 21 and I. 16), every chord of a circle subtends at any point on the circle an angle greater than at any point outside and less than at any point inside the circle, or conversely, according as the lesser or greater angular interval between the containing lines (24) is the subject of comparison for each angle; and, when a circle touches a line, circle, or any other figure, while the point of contact is common to the circle and figure, those at both sides of it on the figure are either both outside or both inside the circle according as the contact of the former with the latter is external or internal, and therefore &c.

The problem "*to find the points on a given line, circle, or any other geometrical figure which subtend maxima or minima angles at two given points*" is reduced, therefore, to the problem "*to describe a circle passing through the two given points and touching the given line, circle, or other figure;*" the solutions of which for the line and circle are respectively as follows:

For the line. If P and Q be the points and MN the line (figs. α and β, Art. 12); describing any circle $PQXY$ passing through P and Q and intersecting or not intersecting MN, and drawing to it a tangent OT from the point O in which the line PQ intersects MN, the circle round O as centre whose radius $= OT$ intersects MN in the points of contact A and B of the two circles required.

For, from the described circle $PQXY$ (Euc. III. 36), $OT^2 = OP.OQ$, and, by construction, OA^2 and OB^2 each $= OT^2$, therefore OA^2 and OB^2 each $= OP.OQ$, and therefore (Euc. III. 37) the circles PQA and PQB touch respectively at A and B the given line MN.

For the circle. If P and Q be the points and MN the circle; describing any circle $PQXY$ passing through P and Q



and intersecting MN in two points X and Y, and from the point O in which the chord of intersection XY meets the line PQ drawing the two tangents OA and OB to MN, their points of contact A and B are those of the two circles required.

For, from the given circle, OA^2 and OB^2 each $= OX.OY$ (Euc. III. 36), and from the described circle $OX.OY = OP.OQ$, therefore OA^2 and OB^2 each $= OP.OQ$, and therefore (Euc. III. 37) the circles PQA and PQB touch respectively at A and B the given circle MN.

If either of the points P or Q were on the line or circle MN, the other not being on it, the two points A and B would evidently coincide with it and with each other; and if P and Q were at opposite sides of the line or circumference MN, A and B would evidently be both impossible as no circle passing through P and Q could then possibly touch MN.

Hence, for the line or circle alike, the two solutions of the problem would be *distinct* if P and Q were at the same side of MN, *coincident* if either P or Q were upon MN, and *impossible* if P and Q were at opposite sides of MN.

52. With respect to the point O, determined as above in the solution for the circle, the following property is important—

The extremities X' and Y' of every chord of MN whose direction passes through O lie in the same circle with P and Q, and conversely, the chord of intersection $X'Y'$ of every circle passing through P and Q and meeting MN passes through O.

For, in the first case, the rectangles $OX'.OY'$ and $OP.OQ$ being each equal to the rectangle $OX.OY$ are equal to each other, and therefore &c.; and, in the second case, conceiving O connected with either point of intersection X' of the two circles MNX' and PQX', and supposing the connecting line OX' to meet them again if possible at two different points Y' and Y'', we would have the two different rectangles $OX'.OY'$ and $OX'.OY''$ equal to the same rectangle $OP.OQ$ which could not be, and therefore &c.

Hence the general property that—

If a variable circle pass through two fixed points P and Q and intersect a fixed circle MN, the variable chord of intersection XY passes through a fixed point O on the line PQ, that, viz., for which the constant rectangle $OX.OY =$ the fixed rectangle $OP.OQ$.

The circle MN being given, if P and Q be both given, O is of course implicitly given with them, being, as above, the point in which XY (the chord of intersection with MN of any circle through P and Q) meets PQ; but, if on the other hand, O only be given, P and Q may be (as in 12) on any line passing through O, and at any two distances from O (measured in similar or opposite directions according as O is external or internal to MN) for which $OP.OQ =$ the given rectangle $OX.OY$.

53. The problem to describe a circle passing through two given points P and Q and touching a given line or circle MN, is evidently a particular case of the problem,

To describe a circle passing through two given points P and Q and intercepting on a given line or circle MN a segment or chord of given length XY.

To solve which, as the direction of XY passes, by the preceding, in either case through O, we have $OX.OY = OP.OQ$ and $OX \mp OY = XY$ according as P and Q are at similar or opposite sides of MN, therefore, by Euc. II. 6 or 5, we have OX and OY and therefore X and Y themselves.

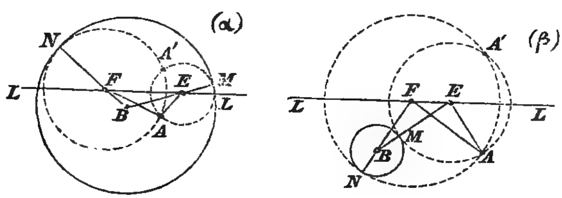
When P and Q are at the same side of MN, any length of segment or chord XY (less of course than the diameter in the case of the circle) might be intercepted by a circle through

P and Q, but when P and Q are at opposite sides of MN, figs. β Art. 12 and γ Art. 51, since the rectangle under the segments of a line cut internally can never (Euc. II. 5) exceed the square of half the line, no length less than twice the side of the square = the rectangle $OP.OQ$ could be intercepted; in that case, therefore, the two solutions of the problem are, distinct for any greater length, coincident for that particular length, and impossible for any lesser length.

Cor. Since a circle passing through a fixed point and having its centre on a fixed line passes necessarily through a second fixed point the reflexion of the first with respect to the line (50), the four following problems are reduced immediately to the preceding.

To describe a circle passing through a given point, having its centre on a given line, and touching, or intercepting a given segment or chord of, a given line or circle.

54. If A be any point, A' its reflexion with respect to any line L, and E and F the centres of the two circles passing through A and A' and touching any circle MN, figs. α and β,



then, if B be the centre of MN, it is evident that $AE+BE$ and $AF+BF$ in fig. α, and $AE \sim BE$ and $AF \sim BF$ in fig. β = the radius BM or BN of MN. Hence the following solutions of the two useful problems—

On a given line L to determine the two points E and F, the sum or difference of whose distances from two given points A and B shall be given.

With either of the two given points B as centre and with a radius BM or BN = the given sum (fig. α) or difference (fig. β) describe a circle MN, the centres E and F of the two circles passing through the other given point A and its re-

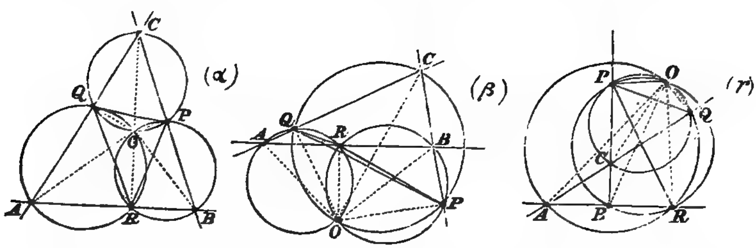
flexion A' with respect to the given line L and touching that circle, are the two points required.

Should MN happen to pass through either A or A' the two points of contact M and N would evidently coincide at whichever of them it passed through; therefore the two centres E and F would also coincide, and the construction *then at the extreme limit of possibility or impossibility* would become that already given in (49) for the *minimum* sum and *maximum* difference of the distances in question.

55. The next example, again, we give separately as leading naturally to an important property of similar figures.

a. Of all triangles of any constant species, whose sides pass through three fixed points, the maximum is that the perpendiculars to whose sides at the points intersect at a common point.

b. Of all triangles of any constant species, whose vertices lie on three fixed lines, the minimum is that the perpendiculars to the lines at whose vertices intersect at a common point.



For if ABC and PQR be any two triangles such that the sides of ABC pass through the vertices of PQR, or the vertices of PQR lie on the sides of ABC; the three circles QAR, RBP, PCQ pass evidently in all cases (Euc. III. 21, 22) through a common point O (for which the three angles QOR, ROP, POQ are equal or supplemental to the three angles BAC, CBA, ABC respectively, and the three angles BOC, COA, AOB to the sums or differences of the three pairs of angles BAC and QPR, CBA and RQP, ACB and PRQ respectively (see 24), and which, when either of the two triangles ABC or PQR is fixed and the species of the other constant, is therefore fixed, and determines with the three sides of the variable triangle, which-

ever it be, three variable triangles BOC, COA, AOB, or QOR, ROP, POQ of constant species revolving round it as a common vertex. Hence, O being fixed in both cases, when, as in (a), PQR is fixed and ABC variable, BC, CA, AB are maxima with OA, OB, OC, that is, when the latter are diameters of the three fixed circles QOR, ROP, POQ respectively, and therefore &c.; and when, as in (b), ABC is fixed and PQR variable, QR, RP, PQ are minima with OP, OQ, OR, that is, when the latter are perpendiculars to the three fixed lines BC, CA, AB respectively, and therefore &c.

Hence, *to construct the triangle of given species and maximum area* ABC *whose sides shall pass through three given points* PQR, *or the triangle of given species and minimum area* PQR *whose vertices shall lie on three given lines* BC, CA, AB. The three angles QOR, ROP, POQ in the former case, and the three BOC, COA, AOB in the latter, being given by the above relations, the point O therefore in either case is given immediately by the common intersection of three given circles (Euc. III. 33), and therefore the three perpendiculars BC, CA, AB to OP, OQ, OR in the former case, and the three OP, OQ, OR to BC, CA, AB in the latter, are given, and therefore &c.

Cor. 1°. By aid of the point O, determined as above, the two problems: *to construct a triangle* ABC *or* PQR *of given magnitude and species, whose three sides* BC, CA, AB *shall pass through three given points* P, Q, R, *or whose three vertices* P, Q, R *shall lie on three given lines* BC, CA, AB, of which the two above are the extreme cases, may be solved with equal readiness; for, the species of the six triangles BOC, COA, AOB and QOR, ROP, POQ being given in both cases, when, as in the former case, the three lengths BC, CA, AB are given, so therefore are the three OA, OB, OC, and therefore the three points A, B, C on the three given circles QOR, ROP, POQ, and when, as in the latter case, the three lengths QR, RP, PQ are given, so therefore are the three OP, OQ, OR, and therefore the three points P, Q, R on the three given lines BC, CA, AB.

Hence, again, as in the problems, Arts. 51 and 53, the two solutions of the problem are distinct, coincident, or im-

possible according as the given magnitude of the triangle to be constructed ABC or PQR, is less than, equal to, or greater than its maximum value in the former case, or greater than, equal to, or less than its minimum value in the latter.

Cor. 2°. By aid of the same again the two problems: *to construct a quadrilateral of given species, whose four sides A, B, C, D shall pass through four given points P, Q, R, S, or whose four vertices P, Q, R, S shall lie on four given lines* A, B, C, D may be readily solved. For, in the former case, to find any vertex AB of the required quadrilateral $ABCD$. As the two triangles PRQ and PSQ, through whose common vertices P and Q the two sides A and B corresponding to that vertex pass, are given, and as the two triangles ACB and ADB, which they determine with the other two sides C and D, are of given species; therefore by the above the circle passing through P and Q and though the required vertex AB passes through two given points M and N, whose distances from AB have a given ratio and which therefore determine AB. And, in the latter case, to find any side PQ of the required quadrilateral $PQRS$, as the two triangles ACB and ADB, on whose common sides A and B the two vertices P and Q corresponding to that side lie, are given, and as the two triangles PRQ and PSQ which they determine with the other two vertices R and S are of given species; therefore, by the above, the circle passing through the intersection of A and B and though the extremities of the required side PQ passes through two given points M and N, which consequently determine that circle and with it therefore the two points P and Q at which it intersects the two given lines A and B.

The same problem, the solutions of which, differing from those of Cor. 1°, are always in both cases unique and possible, may also, in the former case, to which the latter is evidently reducible, be solved otherwise thus as follows: Since the diagonal connecting any pair of opposite vertices AB and CD of the required quadrilateral $ABCD$ divides the two corresponding angles AB and CD each into segments of given magnitude; it therefore intersects the two given circles through P and Q and through R and S, on which AB and CD lie, at two given points I and J which consequently determine that diagonal and therefore the quadrilateral.

N.B. If the two points M and N in the former or the two I and J in the latter of the constructions just given *happened to coincide*, the construction otherwise determinate would be evidently indeterminate, and consequently *an infinite number of quadrilaterals could be constructed satisfying the conditions of the problem.* The circumstances under which such cases arise in general will be considered further on.

Cor. 3°. In the particular case of the above when, as is nearly the case in fig. β, one angle of the triangle $PQR =$ two right angles, and when therefore the other two each $= 0$, it is evident from the values of the three angles BOC, COA, AOB, as given above, that the point O lies on the circle circumscribing the triangle ABC. Hence we see that—

a. If three points P, Q, R, taken arbitrarily on the three sides BC, CA, AB of any triangle ABC, lie in a right line; the three circles QAR, RBP, PCQ intersect at a common point O on the circle ABC.

b. If while the triangle is fixed the three points P, Q, R vary so as to preserve the constancy of the three ratios $QR : RP : PQ$, the point of intersection O is a fixed point, and conversely.

The four lines BPC, CQA, ARB and PQR, in the above, being entirely arbitrary, it follows at once from property a, as the reader may very easily prove *à priori* for himself, that—

The four circles circumscribing the four triangles determined by any four arbitrary lines taken three and three intersect at a common point.

By Cors. 1° and 2° applied to the same particular case we obtain ready solutions of the two following problems, viz.

1°. *To draw a line intersecting three given lines so that its segment intercepted between any two of them shall be cut in given lengths by the third.*

2°. *To draw a line intersecting four given lines so that its segment intercepted between any two of them shall be cut in given ratios by the other two.*

56. From the nature of similar figures and of their homologous points and lines, it appeared (40) that if one point O of or connected with a figure F of any nature variable in magnitude and position but invariable in form be fixed, all points

P, Q, R, S, &c. of or connected with it describe, and all lines A, B, C, D, &c. of or connected with it envelope similar figures, so that in such a case if one point P move on a line or describe a circle, all points Q, R, S, &c. move on lines or describe circles, and if one line A turn round a point or envelope a circle, all lines B, C, D, &c. turn round points or envelope circles. Hence from the preceding it follows that—

For a figure F of any nature variable in magnitude and position but invariable in form, if three points P, Q, R connected with it in any manner move on fixed lines A, B, C, all points S, T, &c. connected with it move on fixed lines D, E, &c., and if three lines A, B, C connected with it in any manner turn round fixed points P, Q, R, all lines D, E, &c. connected with it turn round fixed points S, T, &c.

For, in the former case, the variable triangle P, Q, R, whose vertices move on the three fixed lines A, B, C, and in the latter case the variable triangle A, B, C, whose sides pass through the three fixed points P, Q, R, being invariable in form; therefore by the preceding the point O, connected as above with the variable triangle and therefore with the figure, in both cases is a fixed point, and therefore &c.

COR. 1°. The above general properties supply obvious solutions of the four following general problems, viz.

To construct a figure of given form, 1°. *four of whose points shall lie on given lines;* 2°. *four of whose lines shall pass through given points;* 3°. *three of whose points shall lie on given lines, and one of whose lines shall pass through a given point;* 4°. *three of whose lines shall pass through given points, and one of whose points shall lie on a given line.*

Of these four general problems 1°. and 2°. admit always of possible and generally of unique solutions, depending on the unique point of intersection of two lines in 1°., and on the unique line of connection of two points in 2°., which may however by the possible coincidence of the two lines in 1°., or of the two points in 2°. become in certain cases *indeterminate* (55, Cor. 2°.); 3°. and 4°. on the other hand admit in all cases of two solutions, distinct, coincident, or impossible according to circumstances.

The circumstances under which the solutions of 1°. and 2°.

may in certain cases become indeterminate, appear at once from the two general properties of the present article; the four points of the figure P, Q, R, S, in 1°. may so correspond to the four given lines A, B, C, D, or the four lines of the figure A, B, C, D in 2°. to the four given points P, Q, R, S, that when in 1°. three of the points P, Q, R lie on three of the lines A, B, C, the fourth point S must lie on the fourth line D, or when in 2°. three of the lines A, B, C pass through three of the points P, Q, R, the fourth line D must pass through the fourth point S; in either case the problem would evidently admit of an infinite number of solutions and consequently be indeterminate.

Cor. 2°. The same again by aid of the principles established in the preceding article supply obvious solutions of the four following additional problems, viz.—

To construct a figure of given form and of minimum or given magnitude, three of whose points shall lie on given lines.

To construct a figure of given form and of maximum or given magnitude, three of whose lines shall pass through given points.

Of which the two for the cases of given magnitude admit each, as in Cor. 1°. of the preceding, of two solutions, distinct, coincident, or impossible, according as the given magnitude happens to be within, upon, or beyond the limiting value of which it is susceptible under the circumstances of the case.

57. There are many cases in which a variable magnitude is shewn to be a maximum (or a minimum) in some particular relative position of the elements of the figure with which it is connected, by its being shewn that for any other relative position it could be increased (or diminished), and that every change which would increase (or diminish) it would tend to bring it to the particular configuration in question, of this the four following instructive examples may be taken as illustrations:

Ex. 1. The sum of the distances of a variable point on a fixed line from two fixed points at the same side of the line is a minimum when they make equal angles with the line (Ex. a, 49); from this it follows that—

Of all polygons of any order whose vertices in any assigned order lie on fixed lines, that of minimum perimeter is that whose several angles are all bisected externally by the lines on which their vertices lie.

For, supposing any angle of the polygon not to be so bisected, the removal of its vertex to the point at which it would be so bisected, would,

without affecting in any manner the remaining sides of the polygon, diminish the sum of the containing sides, and therefore the entire perimeter of which that sum is a part.

Ex. 2. For the middle point of any arc of a circle: 1°. The sum of the chords of the segments and the area of the triangle they form with the chord of the arc, are both maxima (Ex. 6°, 48); 2°. The perimeter and area of the quadrilateral formed by the tangent with the chord of the arc and the tangents at its extremities are both minima (Ex. 7°, 48); from these it follows that—

1°. *Of all polygons of the same order inscribed in the same circle, that of maximum perimeter and area is the regular.*

2°. *Of all polygons of the same order circumscribed about the same circle, that of minimum perimeter and area is the regular.*

For, supposing any vertex of the polygon in 1°. not to bisect the arc of the circle intercepted between the adjacent two, its removal to the middle point would, *without affecting in any way the remainder of the polygon,* increase both the perimeter and area of the triangle it determines with the chord of the arc, and therefore of the entire figure of which that triangle is a part; and, supposing the point of contact of any side of the polygon in 2°. not to bisect the arc of the circle intercepted between those of the adjacent two, its removal to the middle point would, *without affecting in any way the remainder of the polygon,* diminish both the perimeter and area of the quadrilateral determined by that side with the chord of the arc and the tangents at its extremities, and therefore of the entire figure of which that quadrilateral is a part.

Ex. 3. When a line of any length is cut into two equal parts, the product of the parts is greater, and the sum of their squares less, than if it were cut into any two unequal parts (Ex. 2°, 48); from this it follows that—

1°. *When a line of any length is cut into any number of equal parts, the continued product of all the parts is greater, and the sum of their squares less, than if it were cut in any way into the same number of unequal parts.*

2°. *When a line of any length is cut into any number of parts $a, b, c, d,$ &c. in the ratios of any set of integer numbers $\alpha, \beta, \gamma, \delta,$ &c., the product $a^\alpha . b^\beta . c^\gamma . d^\delta,$ &c. is greater, and the sum $\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} + \frac{d^2}{\delta} +$ &c. is less, than if it were cut in any other way into the same number of parts.*

To prove 1°. Supposing any two of the parts not to be equal, the equable division of their sum would, *without affecting any of the remaining parts,* increase the product and diminish the sum of the squares of those two, and therefore increase the product and diminish the sum of the squares of the entire set.

To prove 2°. Conceiving a subdivided into α equal parts, b into β equal parts, c into γ equal parts, d into δ equal parts, &c., then since

$a^a = \alpha^a$ times the continued product of the α subdivisions of a, $b^\beta = \beta^\beta$ times the continued product of the β subdivisions of b, $c^\gamma = \gamma^\gamma$ times the continued product of the γ subdivisions of c, $d^\delta = \delta^\delta$ times the continued product of the δ subdivisions of d, &c.; and since $a^2 = \alpha$ times the sum of the squares of the α subdivisions of a, $b^2 = \beta$ times the sum of the squares of the β subdivisions of b, $c^2 = \gamma$ times the sum of the squares of the γ subdivisions of c, $d^2 = \delta$ times the sum of the squares of the δ subdivisions of d, &c., therefore $a^\alpha . b^\beta . c^\gamma . d^\delta$. &c. $= \alpha^\alpha . \beta^\beta . \gamma^\gamma . \delta^\delta$. &c. times the continued product of the whole $\alpha + \beta + \gamma + \delta +$ &c. subdivisions, of the entire line, and $\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} + \frac{d^2}{\delta} +$ &c. = the sum of the squares of the same $\alpha + \beta + \gamma + \delta +$ &c. subdivisions; and therefore, by 1°., the former is a maximum and the latter a minimum when the several subdivisions are *all* equal, that is, as a contains α of them, b, β of them, c, γ of them d, δ of them, &c., when $a : b : c : d$, &c. :: $\alpha : \beta : \gamma : \delta$, &c. Q.E.D.

Ex. 4. When two conterminous lines of any lengths are placed at a right angle, the area of the triangle they determine is greater than if they were placed at any other angle obtuse or acute (Ex. 1°., 48) from this it follows that—

1°. *When all the sides but one of a polygon are given in length and order, the area of the figure is the maximum when the semicircle described on the closing side as diameter passes through all its vertices.*

2°. *When all the sides of a polygon are given in length and order, the area of the figure is the maximum when all its vertices lie in a circle.*

3°. *When the extremities of a bent line of given length are connected by a straight line, the area of the enclosed figure is the maximum when its form is a semicircle.*

4°. *When the perimeter of a closed figure is given, its area is the maximum when its form is a circle.*

To prove 1° and 3°. Supposing any single vertex P of the polygon in the former case, or any single point P of the bent line in the latter case, not to lie upon the semicircle described on the closing side or connecting line AB, then the two conterminous lines AP and BP not being at a right angle (Euc. III. 31). The putting of them at a right angle would, *without affecting in any way, except in position, the remaining portions of the figure which might be regarded as attached to and moveable with them,* increase the area of the triangle APB and therefore of the entire figure of which it is a part.

To prove 4°. Supposing the perimeter to form a circle, then any diameter AB would divide the whole figure into two semicircles, one or both of which would necessarily be altered in form and therefore diminished in area (3°.) by any change whatever from the circular form of the entire.

To prove 2°. Supposing the several vertices of the polygon to lie in a circle, then conceiving the circle described through them, any change whatever in the figure of the polygon would *without affecting in any way except in position the circular segments on the several sides which might be*

regarded as attached to and moveable with them alter the form and therefore diminish the area of the entire circular figure (4°.), and consequently of the polygon itself the only part of the whole undergoing change of area.

Otherwise thus, from 1°. without the aid of 4°. the polygon and circle being supposed described as before; then, firstly, if any two vertices of the former A and B happened to determine a diameter of the latter, that diameter would divide the polygon into two whose areas, by 1°., would both be diminished by any change of figure they could receive; and, secondly, if no two vertices happened to determine a diameter, then drawing any diameter AB, and connecting its extremities A and B each with the two adjacent vertices of the polygon M and N, P and Q, between which it lies, that diameter would divide the entire figure consisting of the variable original polygon and the two invariable appended triangles MAN and PBQ into two parts, whose areas, by 1°., would both be diminished by any change of figure they could receive; therefore in either case any change of figure in the original polygon, as necessarily producing a change of figure in one or both of the partial polygons, would diminish the area of one or both, and therefore of the whole.

The former demonstration, though perhaps less elementary, will probably be regarded by the reader as simpler than the latter.

58. In the Theory of Maxima and Minima it happens very often, so often as to require special notice at the very outset of the subject, that a variable magnitude which in a certain relative position of the elements of the figure with which it is connected has a maximum and a minimum value each for the proper position corresponding to itself, appears in another relative position of the very same elements to have two maxima or two minima values for the same positions alternating with two minima values each $=0$, or two maxima values each $=\infty$, at certain intermediate positions, as, for example, the distance of a variable point on a fixed circle from a fixed line, which when the circle and line do not intersect, is a maximum for the farther and a minimum for the nearer extremity of the diameter perpendicular to the line, but which when they do intersect has apparently maxima values at both those extremities alternating with apparently minima values each $=0$ at the two points of intersection.

In the preceding, and in all similar cases, however—as will more fully appear when we come to the subject of the *Signs* of geometrical magnitudes—*a change of sign takes place at each passage of the variable through* 0 *or* ∞, after which a

negative increase is of course a positive decrease, and conversely, and a negative maximum consequently a positive minimum, and conversely; and the two values $=0$ or ∞ are not real *minima* or *maxima* values at all (45), but merely *the particular values through which the variable magnitude in continuous decrease or increase passes at the moment of changing sign.* Of course if absolute values of magnitudes only were taken into account, as in arithmetic and in the geometry of ancient times, the particular values 0 and ∞ would be the least of all minima and the greatest of all maxima for magnitudes of every kind; but in the geometry of the present day, in which magnitudes of certain kinds are regarded as having not only absolute value but also sign, they are looked on as in no way differing from any other particular values through which variable magnitudes in continuous decrease or increase may happen to pass. In the case of magnitudes incapable of change of sign, the values 0 and ∞ are of course the extreme minima and maxima values in modern as in ancient geometry, and it might at first sight appear questionable whether it would not be better to regard them as such for magnitudes of all kinds as well. The advantages, however, resulting from the convention of signs in modern geometry are so numerous and considerable, that in the present state of the science it could scarcely be regarded as optional to forego them or not.

59. The extreme maxima and minima values of variable magnitudes, in whichever light regarded, give evidently in all cases the extreme *limits of possibility and impossibility* in the solutions of all problems involving the magnitudes; it being of course impossible to construct a magnitude of any kind greater than the extreme maximum or less than the extreme minimum of which it is susceptible under the circumstances of its data.

Should the extreme maxima and minima values of a magnitude variable in position *happen to be equal*, of course all intermediate values would be also equal, and the magnitude would be *constant;* in every such case the problem to construct the magnitude so as to have *a given* value would of course be *impossible* for *any other* than the constant value, while for that

value it would evidently *admit of an infinite number of solutions* or be *indeterminate* as it is termed.

When on the other hand, as is of course the case generally, the extreme maxima and minima values of a magnitude variable in position are not equal, the problem to construct the magnitude so as to have any intermediate value, admits always of at least *two* distinct and definite solutions, more or less separated from each other, which approach to coincidence as the value continuing within the limits of possibility approaches either limit, which actually coincide for each limiting value, and which become impossible together once the limits are passed; and the same is the case generally for all problems admitting of two solutions and therefore for all in which, directly or indirectly, the circle is involved, *the two solutions in general distinct become coincident at the limits of possibility and impossibility, and so pass together through coincidence from possibility to impossibility, and conversely*, (See 21).

As an example of the preceding principles: suppose it were required to draw from a given point to a given circle a line of given length. For the centre of the circle the solutions of the problem would manifestly be impossible for any value of the given length different from the radius and indeterminate for that value; while for every point different from the centre it would admit of two, and but two, determinate solutions which would be distinct, coincident, or both impossible, according as the given length happened to lie between, upon, or beyond the extreme limits for the point.

The above principles are all general and deserving of particular attention; for, 1°.—No problem in geometry admitting in its general form of but a single solution ever becomes impossible, however in certain cases it may appear to do so; 2°.—Whenever a problem admitting in its general form of two solutions becomes impossible, the two solutions always become impossible together, and pass invariably through coincidence in their transition from possible to impossible, and conversely; and 3°.—There is no problem in geometry that does not become indeterminate under certain circumstances of its data.

CHAPTER IV.

ON THE TRIGONOMETRICAL FUNCTIONS OF ANGLES.

60. If from any point P taken arbitrarily on either M of two indefinite lines M and N intersecting at a point O and constituting an angle of any form MN a perpendicular PQ be let fall upon the other line N, the perpendicular determines with the two lines a right-angled triangle PQO whose form it is evident depends only on that of the angle, and every two of whose sides determine two reciprocal ratios which are implicitly given with, and which, reciprocally, implicitly give the form of the angle. The six ratios thus determined from their importance in the science of Trigonometry are termed the *trigonometrical functions* of the angle, and are designated in that science by appropriate names as follows:

1°. The ratio of the perpendicular PQ to the interval PO between its head and the vertex of the angle is termed the *sine* of the angle.

2°. The ratio of the perpendicular PQ to the interval QO between its foot and the vertex of the angle is termed the *tangent* of the angle.

3°. The ratio of the former interval PO to the latter interval QO is termed the *secant* of the angle.

4°. The ratio of the interval OQ to the distance OP is termed the *co-sine* of the angle.

5°. The ratio of the interval OQ to the perpendicular PQ is termed the *co-tangent* of the angle.

6°. The ratio of the distance OP to the perpendicular PQ is termed the *co-secant* of the angle.

Upon the question as to the origin and appropriateness of the names 'sine,' 'tangent,' and 'secant,' we need not enter here; the three simple ratios so designated are of such frequent oc-

currence, the first of them especially, in geometrical researches, as absolutely to require some distinguishing appellations; and the old and familiar names by which they have always been known in another science, are at least as convenient as any others that might be proposed for the purpose; the remaining three, termed respectively co-sine, co-tangent, and co-secant, have been so named as being to *the complement of the angle* what the sine, tangent, and secant, are to the angle itself.

If the angle determined by the two lines be conceived to change figure and to pass continuously through every variety of form, from the extreme of two parallel to the opposite extreme of two rectangular lines; the whole six ratios will pass evidently in the course of the variation, the sine, tangent, and secant in continuous increase, and the co-sine, co-tangent, and co-secant in continuous decrease, through every variety of value of which they are severally susceptible; the sine from 0 up to 1 and the co-sine from 1 down to 0, the secant from 1 up to ∞ and the co-secant from ∞ down to 1, the tangent from 0 up to ∞ and the co-tangent from ∞ down to 0; the whole six being of course implicitly given for each particular form of angle, and any one of them reciprocally determining the corresponding form of the angle and with it of course the remaining five.

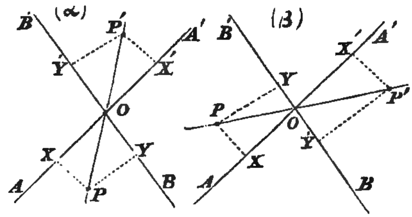
Of all the trigonometrical functions of the angle the *sine* is that which enters most largely into the investigations of modern geometry, and we shall accordingly devote the present chapter to the consideration of a few simple but very important properties involving the sines of angles.

61. *The ratio of the sines of the segments into which an angle is divided by any line passing through its vertex is the same as that of the perpendiculars on its sides from any point on the line; and conversely, the ratio of the perpendiculars from any point on the sides of an angle is the same as that of the sines of the segments into which the angle is divided by the line connecting its vertex with the point.*

For if AA' and BB', or M and N, be the sides of the angle; PP', or L, the line passing through its vertex O; P, or P', the point, and PX and PY, or $P'X'$ and $P'Y'$, the perpendiculars. Then since by definition $PX : PO$ or $P'X' : P'O = \sin LM$

and $PY : PO$ or $P'Y' : P'O = \sin LN$, therefore $PX : PY$ or $P'X' : P'Y' = \sin LM : \sin LN$, and therefore &c.

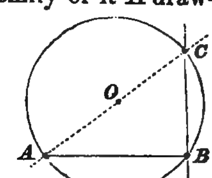
COR. 1°. If the two perpendiculars PX and PY, or $P'X'$ and $P'Y'$, were turned round P, or P', through any common angle so as to become, more generally, isoclinals inclined at any equal angles to the sides of the given angle M and N, the same property would obviously be true of the isoclinals as of the perpendiculars, as the ratio of the former would evidently be constant and equal to that of the latter through whatever angles they were turned.



COR. 2°. A very obvious solution of a very useful problem "*to divide a given angle internally or externally into two parts whose sines shall have a given ratio*" might evidently be based on the above, but another and in many respects more convenient method of effecting the same division will be given further on.

62. *In a circle the ratio of any chord to the diameter is the sine of the constant angle subtended by the chord at every point on the circumference of the circle* (25).

If AB be the chord; through either extremity of it A drawing the diameter AC and joining CB, then since the angle subtended by the chord at any point on the circle is independent as to form of the position of the point (25), if the theorem be true for any one point on the circle it is true for every point, but it is true for the point C, for the angle ABC being in a semicircle and consequently a right angle, therefore by (60) the ratio of $AB : AC$ is the sine of the angle ACB, and therefore &c.



COR. 1°. Hence *two or any number of chords of the same circle are to each other as the sines of the angles they severally subtend at the circumference of the circle;* for each chord, by the above, being equal to the diameter of the circle multiplied

by the sine corresponding to itself, and the diameter being the same for them all, therefore &c.

COR. 2°. The angle any chord of a circle makes with the tangent at either of its extremities being similar in form to that in the two segments into which it divides the circle (22). Hence from Cor. 1°.—

Two or any number of chords of the same circle are to each other as the sines of the angles they make with the tangents at their several extremities.

COR. 3°. The several chords may be conterminous, in which case it appears at once from Cor. 2°., that—

Two or any number of chords diverging from the same point on the circumference of a circle are to each other as the sines of the angles they severally make with the tangent at the point.

COR. 4°. Any two adjacent sides of any polygon inscribed in a circle being conterminous chords of the circle; therefore from Cor. 3°.—

The tangents at the several vertices of any polygon inscribed in a circle divide the several angles of the polygon externally into parts whose sines are in the ratios of the adjacent sides of the polygon.

COR. 5°. The three sides of every triangle being chords of the same circle, that viz. which passes through its three vertices, and the three angles being those subtended by their opposite sides at the circumference of the circle; hence at once, from the above, the important property of the triangle, that—

The sine of any angle of a triangle is equal to the opposite side divided by the diameter of the circle circumscribing the triangle; and conversely, the diameter of the circle circumscribing any triangle is equal to any side of the triangle divided by the sine of the opposite angle.

COR. 6°. Denoting in any triangle by a, b, c the three sides, and by A, B, C the three respectively opposite angles, then always—

$$a \div \sin A = b \div \sin B = c \div \sin C,$$

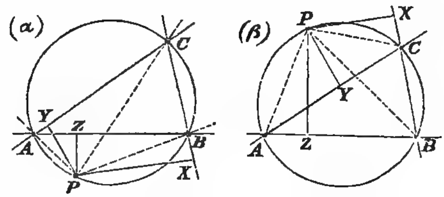
for each by the above is equal to the diameter of the circle circumscribing the triangle.

COR. 7°. Denoting by d the diameter of the circumscribing circle, and by p, q, r the three perpendiculars from the three vertices A, B, C upon the respectively opposite sides a, b, c; then since (60), $p = c.\sin B$ or $b.\sin C$, $q = a.\sin C$ or $c.\sin A$, $r = b.\sin A$ or $a.\sin B$, and since, Cor. 6°., $d = a \div \sin A = b \div \sin B = c \div \sin C$, therefore $pd = bc$, $qd = ca$, $rd = ab$, and therefore generally—

In every triangle the product of any two sides is equal to the product of the diameter of the circumscribing circle into the perpendicular on the third side from the opposite vertex.

This property supplies an obvious method of solving the problem: "*given of a triangle one side, the opposite angle, and the product of the other two sides to construct it.*"

COR. 8°. If P be any point on the circumscribing circle, and PA, PB, PC the three lines connecting it with the three vertices A, B, C, then since, whatever be the position of P, any two of the connecting chords PA and PB, divided each by the diameter of the circle d, are the sines of the two segments PCA and PCB into which the third PC divides, internally or externally, the angle ACB through whose vertex it passes; it follows that—



The two general problems: "to divide a given angle internally or externally into two parts whose sines shall have any given relation to each other," and "to divide a given arc of a circle, internally or externally into two parts whose chords shall have the same relation to each other," are identical.

COR. 9°. If PX, PY, PZ be the three perpendiculars from P on the three sides BC, CA, AB of the triangle ABC, then since, Cor. 7°.,

$$PB.PC = d.PX, \quad PC.PA = d.PY, \quad PA.PB = d.PZ;$$

therefore

$$\sin PAB.\sin PAC = PX \div d,$$
$$\sin PBC.\sin PBA = PY \div d,$$
$$\sin PCA.\sin PCB = PZ \div d;$$

and therefore generally—

The product of the sines of the segments into which any angle of a triangle is divided by a variable line passing through its vertex, varies as the perpendicular to the opposite side from the point in which the line meets the circumscribing circle.

This property gives a very definite conception of the law according to which the product of the sines of the segments of a fixed angle by a variable line of section varies with the position of the dividing line, and supplies moreover an obvious solution of the useful problem—

To divide a given angle internally or externally into two parts whose sines shall have a given product.

Cor. 10°. If A, B, C, D be any four points on a circle, PX and PX', PY and PY', PZ and PZ' the three pairs of perpendiculars from any fifth point P on the circle upon the three pairs of opposite chords BC and AD, CA and BD, AB and CD they determine, and d the diameter of the circle; then since by Cor. 7°.,

$$PX = PB.PC \div d \text{ and } PX' = PA.PD \div d,$$
$$PY = PC.PA \div d \text{ and } PY' = PB.PD \div d,$$
$$PZ = PA.PB \div d \text{ and } PZ' = PC.PD \div d;$$

therefore

$$PX.PX' = PY.PY' = PZ.PZ' = PA.PB.PC.PD \div d^2;$$

and therefore—

The products of the three pairs of perpendiculars from any point on a circle upon the three pairs of opposite chords connecting any four points on the circle are equal; and their common value is equal to the product of the distances of the one point from the four divided by the square of the diameter of the circle.

If the six perpendiculars were turned round the point P through any common angle, so as to become, more generally, isoclinals inclined at any equal angles to the six chords; the products of the three pairs of isoclinals for opposite pairs of chords would still continue equal, each isoclinal being equal to the corresponding perpendicular multiplied by the secant of the angle of rotation.

63. *In every triangle the ratio of the sines of any two of the angles is the same as that of the sides opposite to them.*

For if A, B, C be the three angles, a, b, c the three opposite

sides, and p, q, r the three perpendiculars on the latter from the opposite vertices; since then by definition $\sin B = p : c$ and $\sin C = p : b$, $\sin C = q : a$ and $\sin A = q : c$, $\sin A = r : b$ and $\sin B = r : a$, therefore at once

$$\sin B : \sin C = b : c,\ \sin C : \sin A = c : a,\ \sin A : \sin B = a : b,$$

and therefore generally for all three,

$$\sin A : \sin B : \sin C = a : b : c,$$

or, *in every triangle the sines of the angles are as the opposite sides.*

Otherwise thus: conceiving a circle circumscribed round the triangle, then since, by the preceding (62), each side divided by the diameter of the circle is the sine of the opposite angle, therefore &c.

This latter though less direct has the advantage over the former and more ordinary method of proving this important theorem, that besides establishing the proposition it gives at the same time the common value of the three equivalent quotients $a \div \sin A$, $b \div \sin B$, $c \div \sin C$, viz. *the diameter of the circle circumscribing the triangle.*

Cor. 1°. The angle between any two lines being similar in form to that between parallels to them through any point, it follows at once from the above, that—

Every three lines drawn from a point parallel and equal to the three sides of a triangle are to each other each as the sine of the angle between the other two.

That is, if O be the point, and OA, OB, OC the three lines, then $OA : OB : OC = \sin BOC : \sin COA : \sin AOB$.

Cor. 2°. In every parallelogram any two adjacent sides and the conterminous diagonal being equal and parallel to the three sides of either triangle into which the parallelogram is divided by the diagonal. Hence from Cor. 1°—

Each side of every parallelogram is divided by the diagonal which passes through it into parts whose sines are in the inverse ratio of the adjacent sides of the parallelogram.

That is, if OA and OB be the sides about the angle, and OD the diagonal, then $\sin AOD : \sin BOD = OB : OA$.

Cor. 3°. The above supplies obvious and rapid solutions of the two following problems:

1°. *To divide two or four right angles into three parts whose sines shall be as three given numbers.*

2°. *To determine two angles whose sines shall be to the sine of their sum or difference as two given numbers to a third.*

For, in the case of 1°., constructing any triangle whose three sides are as the three numbers, its three internal angles furnish obviously the solution for two and its three external for four right angles, (Euc. I. 32); and in the case of 2°., constructing any triangle two of whose sides are to the third as the two given numbers to the third, its two internal angles opposite to the two sides furnish obviously the solution for the case of the sum, and either of them with the external adjacent to the other for the case of the difference (Euc. I. 32).

If the three given numbers were such that three lines representing them were incapable of forming a triangle, that is, (Euc. I. 20), if one of them were greater than the sum or less than the difference of the other two, the above constructions would of course fail; thus showing that in such cases the required division or determination would be impossible.

Cor. 4°. The three internal angles BOC, COA, AOB, subtended by the three sides BC, CA, AB of any triangle ABC at any arbitrary point O, being either together equal to four right angles, or each separately equal to the sum or difference of the other two, according as the point O is within or without the triangle; the above leads again, as in Cor. 3°, to the four solutions of the following problem, viz.—

To determine the point O for which the sines of the three angles subtended by the three sides of one given triangle ABC shall be as the three sides of another given triangle $A'B'C'$.

For the three angles subtended at O by the three sides of ABC being, according to the position of O, as just observed, either the three external or one of the external and two of the internal corresponding angles of $A'B'C'$; therefore describing on the three sides of ABC as chords the three pairs of equal circles which intersect them internally and externally at the three corresponding angles, internal and external, of $A'B'C'$ (22); of the six circles thus described, the three which intersect the sides of ABC internally at the three internal and externally at the three external corresponding angles of $A'B'C'$ intersect with each other at a common point O, which is one of those required, and intersect with the remaining

three, each with the two not corresponding to itself, at three other common points P, Q, R, which are the remaining three of those required.

In the particular case when the three pairs of corresponding angles of the two triangles ABC and $A'B'C'$ are equal, and when the triangles themselves are therefore similar; while the three circles determining the point O intersect at the point of concurrence of the three perpendiculars from the vertices on the opposite sides of ABC, that being the point for which the three internal angles BOC, COA, AOB, are the supplements of the three internal angles BAC, CBA, ACB, of the triangle; the remaining three evidently coincide with each other and with the circle circumscribing ABC, and the three points P, Q, R are consequently indeterminate. This is also evident *a priori* from (62); every point on the circle circumscribing any triangle ABC subtending, as there shewn, its three sides at angles whose sines are proportional to their lengths.

Cor. 5°. Denoting by P, Q, R the radii of the three equal pairs of conjugate circles in the preceding corresponding to the three sides BC, CA, AB, respectively of the triangle ABC; since then three of those circles for different sides pass through the point O, and consequently circumscribe the three partial triangles BOC, COA, AOB, therefore by (62).

$$2P = BC \div \sin BOC,\ 2Q = CA \div \sin COA,\ 2R = AB \div \sin AOB,$$

and therefore

$$P : Q : R = BC \div \sin BOC : CA \div \sin COA : AB \div \sin AOB.$$

Hence again, by Cor. 4°., the four solutions of the following additional problem, viz.—

To determine the point O such that for three given points A, B, C the radii P, Q, R of the three circles BOC, COA, AOB, shall be as three given numbers.

For since from the propositions just stated

$$\sin BOC : \sin COA : \sin AOB = BC \div P : CA \div Q : AB \div R,$$

the problem is therefore reduced at once to that of Cor. 4°., the two groups of three ratios $BC : CA : AB$ and $P : Q : R$ being both given, and therefore with them the group to which the three sines are proportional.

In the particular case, when $P = Q = R$, since then

$$\sin BOC : \sin COA : \sin AOB = BC : CA : AB,$$

the point O as already noticed in Cor 4°., is either the unique point of concurrence of the three perpendiculars from the vertices on the opposite sides, or any point indifferently on the circumscribing circle, of the triangle ABC; the three equal circles BOC, COA, AOB, in the former case being equal to, and in the latter case coinciding with, the circle ABC.

64. *In every triangle the ratio of double the area to the rectangle under any two of the sides is the sine of the angle contained by those sides.*

For if, as in the preceding, A, B, C be the three angles, a, b, c the three opposite sides, and p, q, r the three perpendiculars on the latter from the opposite vertices; since then (Euc. I. 42.) 2 area $= ap = bq = cr$, and since (60) $p = b \,.\, \sin C$ or $c \sin B$, $q = c \sin A$ or $a \sin C$, $r = a \sin B$ or $b \sin A$, therefore

$$2 \text{ area} = bc \,.\, \sin A = ca \,.\, \sin B = ab \,.\, \sin C,$$

and therefore &c.

Cor. 1°. Since from the above

$$\text{area} = \tfrac{1}{2}bc \,.\, \sin A = \tfrac{1}{2}ca \,.\, \sin B = \tfrac{1}{2}ab \,.\, \sin C,$$

therefore—

In every triangle the area is equal to half the product of any two of the sides multiplied into the sine of the included angle.

Hence if two sides of a triangle be given, the area varies as the sine of the included angle, has equal values for every pair of supplemental angles, and is the maximum for a right angle.

Cor. 2°. Denoting by R the radius of the circle circumscribing the triangle, then since by (62),

$$\sin A = a \div 2\,R, \sin B = b \div 2R, \sin C = c \div 2\,R,$$

and since by the above, $\sin A = 2$ area $\div\, bc$, $\sin B = 2$ area $\div\, ca$, $\sin C = 2$ area $\div\, ab$, therefore $R = abc \div 4$ area, or—

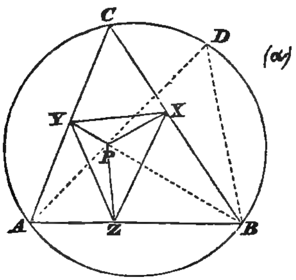
In every triangle the radius of the circumscribing circle is equal to the product of the three sides divided by four times the area.

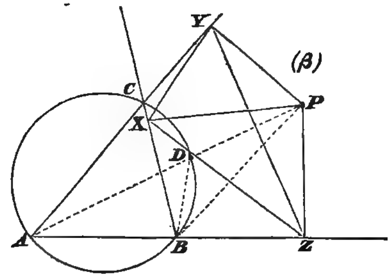
Which is the well-known formula by which to calculate in numbers the value of R, when those of a, b, c are given.

COR. 3°. *If from any point P perpendiculars PX, PY, PZ, be let fall upon the sides BC, CA, AB, of any triangle ABC, then*

$$2 \text{ area of triangle } XYZ = (OR^2 \sim OP^2) \,.\, \sin A \,.\, \sin B \,.\, \sin C$$

where O and OR are the centre and radius of the circle circumscribing the triangle ABC.





For, connecting P with any two of the vertices A and B of the triangle ABC, and the point D where the connector for either A intersects the circumscribing circle with the other B; then by the above, 2 area $XYZ = ZX \,.\, ZY \,.\, \sin XZY$; but the two groups of four points Y, P, Z, A and X, P, Z, B being evidently concyclic, and PA and PB being the diameters of the two circles (Euc. III. 31.); therefore (62), $ZY = PA \,.\, \sin A$, $ZX = PB \,.\, \sin B$, and (Euc. III. 21. 22.) angle $XZY =$ angle PBD, the two angles PZX and PZY being equal to the two PBX and PAY or CBD respectively; therefore

$$2 \text{ area } XYZ = PA \,.\, PB \,.\, \sin A \,.\, \sin B \,.\, \sin PBD,$$

but (63) $PB \,.\, \sin PBD = PD \,.\, \sin PDB = PD \,.\, \sin C$, (Euc. III. 21.) therefore

$$2 \text{ area } XYZ = PA \,.\, PD \,.\, \sin A \,.\, \sin B \,.\, \sin C,$$

and therefore &c; since (Euc. III. 35. 36.) $PA \,.\, PD = (OR^2 - OP^2)$ or $(OP^2 - OR^2)$ according as P is within (fig α) or without (fig β) the circle circumscribing ABC.

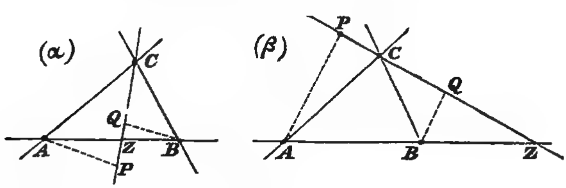
If in the above the three perpendiculars PX, PY, PZ, were turned round P in the same direction of rotation through any common angle, so as to become, more generally, isoclinals PX', PY', PZ' inclined at the complement of the angle to the sides; the same value multiplied by the square of the secant of the angle of rotation, or, which is the same thing, divided by the

square of the sine of the angle of inclination to the sides (60), would evidently (Euc. VI. 19.) be the value of the area of the triangle $X'Y'Z'$.

COR. 4°. It follows of course from the preceding, Cor. 3°., that, whether for perpendiculars or isoclinals, *the area of the triangle XYZ is*—1°. *constant, when P is on any circle concentric with O;* 2°, *evanescent when P is on the circle circumscribing ABC;* 3°, *a maximum in absolute value* (58) *when P is at O, or at infinity;* from the second of which it appears that, *the feet of the perpendiculars upon the sides of a triangle from any point on its circumscribing circle, or more generally of any isoclinals inclined to the perpendiculars at the same angles and in the same directions of rotation, lie in a line;* a property the reader may easily prove directly for himself. See figs. Cor. 9°. Art. 62.

COR. 5°. The preceding properties, Cor. 4°., supply obvious solutions of the three following problems:—"*On a given line or circle to determine the point or points from which if perpendiculars be let fall upon three given lines the area of the triangle determined by their feet shall be a minimum, a maximum, or given;*" or more generally of the three corresponding problems in which the perpendiculars are replaced by isoclinals inclined to them in either direction at any given angle of rotation.

65. *Every line passing through any vertex of a triangle divides the opposite side into segments in the ratio compounded of that of the conterminous sides and of that of the corresponding segments into which it divides the angle at the vertex.*



For, if ABC be the triangle, C the vertex, and CZ the line; letting fall upon CZ from the other two vertices A and B, the two perpendiculars AP and BQ, then since by similar triangles $AZ : BZ = AP : BQ$, and since by (60),

$$AP = AC.\sin ACP = AC.\sin ACZ,$$

and $$BQ = BC.\sin BCQ = BC.\sin BCZ;$$

therefore $$AZ : BZ = AC.\sin ACZ : BC.\sin BCZ,$$

that is, Euc. VI. (23) = the ratio compounded of the two ratios $AC : BC$ and $\sin ACZ : \sin BCZ$, and therefore &c.

Otherwise thus, since by triangles having the same altitude, $AZ : BZ = \text{area } ACZ : \text{area } BCZ$, and since by (64)

$$\text{area } ACZ = \tfrac{1}{2} AC.\,CZ.\sin ACZ,$$

and $$\text{area } BCZ = \tfrac{1}{2} BC.\,CZ.\sin BCZ,$$

therefore as before,

$$AZ : BZ = AC.\sin ACZ : BC.\sin BCZ,$$

and therefore &c.

COR. 1°. If the sides AC and BC about the vertex be equal, then $AZ : BZ = \sin ACZ : \sin BCZ$, or—

Every line passing through the vertex of an isosceles triangle divides the base into segments whose ratio is the same as that of the sines of the segments into which it divides the vertical angle.

COR. 2°. If CZ bisect the angle through whose vertex it passes either internally or externally, then, as in either case $\sin ACZ = \sin BCZ$, therefore $AZ : BZ = AC : BC$, or (Euc. VI. 3)—

The line bisecting internally or externally any angle of a triangle divides the opposite side internally or externally into segments in the ratio of the conterminous sides.

COR. 3°. If CZ divide the angle through whose vertex it passes into segments whose sines are in the inverse ratio of the adjacent sides, that is, so that $\sin ACZ : \sin BCZ = BC : AC$, then $AZ : BZ = 1$, or—

The line dividing internally or externally any angle of a triangle into segments whose sines are in the inverse ratio of the adjacent sides bisects internally or externally the opposite side.

COR. 4°. If CZ divide the angle through whose vertex it passes into segments whose sines are in the direct ratio of the adjacent sides, that is, so that $\sin ACZ : \sin BCD = AC : BC$, then $AZ : BZ = AC^2 : BC^2$, or—

The line dividing internally or externally any angle of a triangle into segments whose sines are in the direct ratio of the adjacent sides divides internally or externally the opposite side into segments in the duplicate ratio of the conterminous sides.

Cor. 5°. As each angle of a triangle is divided externally into segments similar in form to the other two angles both by the parallel through its vertex to the opposite side (Euc. I. 32), and by the tangent at its vertex to the circumscribing circle (Euc. III. 32), the sines of the segments are therefore by (63), inversely in the former case and directly in the latter, in the ratio of the adjacent sides, and therefore, by Cors. 3° and 4° above—

Each side of a triangle is bisected externally by the parallel to it through the opposite vertex, and divided externally into segments in the duplicate ratio of the conterminous sides by the tangent to the circumscribing circle at the opposite vertex.

Cor. 6°. Of the many methods of effecting the very useful division "*to divide a given angle internally or externally into two parts whose sines shall have a given ratio,*" the following based on the above is perhaps on the whole the most convenient.

Connecting any two points A and B taken arbitrarily one on each side of the given angle ACB, (see figures) and cutting the connecting line AB (Euc. VI. 10), internally or externally as the case may be, in the ratio compounded of the known ratio of $AC : BC$ and of the given ratio of the required segments, the line CZ connecting the point of section Z with the vertex of the angle C divides by the above the angle as required.

The two points A and B being both arbitrary, they might be taken so that $AC = BC$, in which case Z would be simply the point of section, internal or external, of AB in the given ratio of the sine $ACZ : \sin BCZ$ (Cor. 1° above), or they might be taken so that $AC : BC$ in the inverse of the given ratio of $\sin ACZ : \sin BCZ$, in which case Z would be simply the point of bisection, internal or external, of AB (Cor. 3° above).

66. *The difference of the squares of the sines of any two angles is equal to the product of the sines of the sum and of the difference of the angles.*

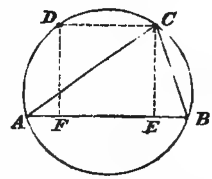
The product of the sines of any two angles is equal to the

difference of the squares of the sines of half the sum and of half the difference of the angles.

The reader familiar with the Second Book of Euclid will at once perceive that these are not two different propositions, but only two different modes of stating the same general property respecting the equal and unequal divisions of an angle; nor can he fail to observe at the same time the complete analogy between the common property they express, and the general property respecting the equal and unequal divisions of a line contained in propositions 5 and 6 of that Book.

On account of their importance, however, we shall give separate and independent demonstrations of each.

To prove the first. Constructing a triangle ABC, two of whose angles A and B are equal to the two angles, and through the third vertex C drawing the chord CD of its circumscribing circle parallel to the opposite side AB; then since $AD = BC$ (Euc. III. 30) and therefore $AC \sim BC = CD$ the four chords AC, BC, AB, CD divided each by the diameter of the circle are respectively (62) the sines of the four angles B, A, $B+A$, $B \sim A$, and to prove the theorem it remains only to shew that $AC^2 \sim BC^2 = AB \cdot CD$.

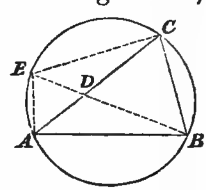


From C and D letting fall CE and DF perpendiculars on AB, then (Euc. I. 47),

$$AC^2 \sim BC^2 = AE^2 \sim BE^2 = (AE + BE) \cdot (AE \sim BE)$$
$$= AB \cdot EF = AB \cdot CD,$$

and therefore &c.

To prove the second. Constructing as before a triangle ABC, two of whose angles A and B are equal to the two angles, measuring from its third vertex C on either of the opposite sides CA a length CD equal to the other CB, joining BD meeting the circumscribing circle of the triangle at E, and drawing AE and CE; then, the angles CBE and ABE being respectively half the sum and half the difference of the angles B and A, the four chords AC, BC, CE, AE divided each by the diameter of the circle are re-



spectively (62) the sines of the four angles B, A, $\frac{1}{2}(B+A)$, $\frac{1}{2}(B \sim A)$, and to prove the theorem it remains only to shew that $AC.BC = CE^2 \sim AE^2$.

The triangle BCD being isosceles by construction, so is the triangle AED which is similar to it (Euc. III. 21), therefore, (Euc. II. 5, 6, Cor.), $EC^2 \sim EA^2 = CA . CD = CA . CB$.

Cor. 1°. The preceding furnish obvious solutions of the two following problems:

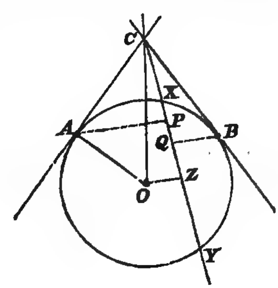
1°. *To divide a given angle, internally or externally, so that the difference of the squares of the sines of the segments shall be given.*

2°. *To divide a given angle, internally or externally, so that the product of the sines of the segments shall be given.*

Cor. 2°. The following deduction from the above furnishes a convenient mode of representation, as well as a very definite conception, of the law according to which the product of the sines of the segments of an angle varies with the change of position of its line of section.

If a circle of any radius be inscribed in an angle the product of the sines of the segments into which the angle is divided by a variable line passing through its vertex varies as the square of the segment of the line intercepted by the circle.

Let O be the centre of the circle, A and B its points of contact with the sides of the angle, and XY the line passing through C; then letting fall OZ perpendicular from O on XY, we have by the above



$$\sin ACZ . \sin BCZ = \sin^2 OCA \sim \sin^2 OCZ$$
$$= (OA^2 \sim OZ^2) \div OC^2$$
$$= (OX^2 \sim OZ^2) \div OC^2$$
$$= XZ^2 \div OC^2 = XY^2 \div 4OC^2 :$$

therefore $\propto XZ^2$ or XY^2. Q.E.D.

When the variable line of section in the course of its revolution round C enters the supplemental region of the angle ACB, the circle AOB is of course no longer available for the above representation; but then it may be replaced by another $A'O'B'$ inscribed in the supplemental region, and the new circle

will continue to represent the law of the variation on the same scale as before, provided only the distance CO' of its centre from the vertex of the angle is equal to the distance CO of the centre of the original circle from the same.

Cor. 3°. Letting fall AP and BQ perpendiculars from A and B on XY; then, since $AP.BQ \div AC.BC = \sin ACZ.\sin BCZ$, therefore, from Cor. 2°., $AP.BQ \div XZ^2 = AC.BC \div OC^2$, or

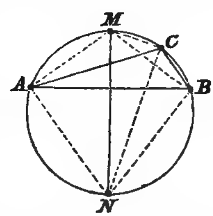
$$4\,AP.BQ \div XY^2 = AC.BC \div OC^2,$$

a property of the circle which may be easily proved directly.

67. *The sum of the sines of any two angles is equal to twice the product of the sines of half the sum and of the complement of half the difference of the angles.*

The difference of the sines of any two angles is equal to twice the product of the sines of half the difference and of the complement of half the sum of the angles.

Constructing, as in the properties of the preceding article, a triangle ABC, two of whose angles A and B are the two angles, bisecting internally or externally the arc ACB of the circumscribing circle at M and N respectively, and connecting both points of bisection with A, B, and C; then the angles MNA, or MNB, and MNC being respectively half the sum and half the difference of the angles CNA and CNB, that is, of the angles B and A, the four chords CA, CB, MA, or MB, and MC divided each by the diameter of the circle are respectively the sines of the four angles B, A, $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A \sim B)$, and the two chords NA, or NB, and NC divided each by the diameter are the sines of the complements of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A \sim B)$; and to prove the theorems it remains only to shew that,



$$(CA + CB) : CN :: (MA + MB) : MN,$$

and that $$(CA \sim CB) : CM :: (NA + NB) : NM.$$

From the two inscribed quadrilaterals $MNCA$ and $MNCB$, since by Ptolemy's theorem,

$$CA.MN = CN.MA \pm CM.NA$$

and $$CB.MN = CN.MB \mp CM.NB,$$

therefore by addition and subtraction

$$(CA + CB) . MN = CN . (MA + MB)$$

and $$(CA \sim CB) . MN = CM . (NA + NB),$$

and therefore &c.

Or, more directly, from the two inscribed quadrilaterals $ABCN$ and $ABCM$, since by the same theorem

$$CA . NB + CB . NA = CN . AB,$$

and $$CA . MB \sim CB . MA = CM . AB,$$

therefore at once

$$(CA + CB) : CN = AB : (AN \text{ or } BN) = (MA + MB) : MN,$$

and $$(CA \sim CB) : CM = AB : (AM \text{ or } BM) = NA + NB : NM,$$

and therefore &c.

Cor. The preceding supply evident solutions of the four problems:

To divide a given angle, internally or externally, into two parts whose sines shall have a given sum or difference.

And the proportions on which they depend of the two problems.

Given of a triangle (ACB) *the base, the vertical angle, and the sum or the difference of the sides, to construct it.*

CHAPTER V.

ON THE CONVENTION OF POSITIVE AND NEGATIVE IN GEOMETRY.

68. THE most striking characteristic of modern as contrasted with ancient geometry is comprehensiveness of language and demonstration. General enunciations on the one hand, and general demonstrations on the other, comprehending in the geometry of the present day all the different cases of the various properties considered, arising from variations in number, position, or magnitude, among the elements of the figures involved, which in the geometry of former days would have been regarded as so many distinct propositions, requiring each a separate statement and independent proof of its own. All such enunciations and demonstrations, moreover, unencumbered, in consequence of this very character of comprehensiveness and generality, with the accidental peculiarities and unessential details of particular cases, and involving accordingly the essential elements of abstract principles only, being thus the more readily apprehended, easily remembered, and instructively suggestive, in proportion as they are comprehensive and general. These important and characteristic advantages are mainly due to the employment, now universally recognised by geometers, of the algebraic signs + and − to indicate the directions in which the various magnitudes coming under their consideration are measured, with regard to which they have laid down the following general rule of convention.

In every case of the comparison of magnitudes susceptible of measurement in either of two opposite directions the signs + and − are employed to distinguish between the directions.

Segments measured on the same line, arcs measured on the same circle, angles measured round the same vertex, triangles

or parallelograms described on the same base, perpendiculars or any other isoclinals erected to or let fall upon the same line, are obvious examples of different kinds of geometrical magnitudes coming under the above head; every two of each of which, when considered together in any number, are therefore to be regarded as having similar or opposite signs according as the directions in which they are measured are similar or opposite.

69. In every application of the above principle of convention it is optional which of the two opposites is to be regarded as the positive and which the negative direction, but the selection once made, and either sign given to either direction in any case, the same sign must be given to the same direction and the opposite sign to the opposite direction throughout the entire case. In the comparison of magnitudes whose directions of measurement are not either similar or opposite, such as segments on different lines, triangles or parallelograms on different bases, perpendiculars or isoclinals to different lines, not parallel to each other, the selection for each separate direction and its opposite is also optional; but once made for each in any case must invariably be adhered to throughout the entire case.

It is this distinctive principle of modern as contrasted with ancient geometry, this recognition of magnitudes as having not only absolute or numerical value but also sign determined by application of the above general rule of convention, which has mainly tended to render the language and demonstrations of the former independent of all accidental variations among the component elements of the figures to which they refer.

70. In accordance with the preceding principle the familiar terms "sum" and "difference" are employed in the geometry of the present day with an important modification of their accustomed significations as employed in the geometry of former times, and to the present day in arithmetic, which must be carefully attended to in order to an accurate, and in many cases even an intelligible conception of the true meaning intended to be conveyed by their use, which is as follows:

The term "sum" as employed in arithmetic is used to denote the result of adding together the numerical values of any number of magnitudes taken absolutely without any regard to their

signs, so that there it is always a positive quantity; in geometry, on the contrary, it is applied to the same result with this difference that the signs of the several magnitudes are taken into account in the addition; so that the geometric sum of any number of magnitudes really means the arithmetic sum of all that are positive among them minus the arithmetic sum of all that are negative, and this is what is uniformly meant by the term "sum" as now invariably employed in geometry unless the contrary be expressly stated.

It thus appears that the sum of any number of geometrical magnitudes is to be regarded as positive, negative, or nothing, according as the aggregate of the positive individuals or terms composing it happens to exceed, fall short of, or equal, that of the negative.

All that has been said in the above remarks applies equally to the term "difference" as employed in the geometry of the present day in reference to two magnitudes. It denotes in arithmetic the result of subtracting one from the other attending only to their absolute values, and in geometry the same result taking into account also their signs; thus the geometrical difference of two magnitudes may be their arithmetic sum, and conversely.

71. Similar remarks apply to the terms "product" and "quotient" as employed in the geometry of the present day, compared with their known significations as employed in arithmetic; in the latter, as in the cases of "sum" and "difference," the absolute values of the magnitudes only being taken into account, while in the former their signs also are attended to. Hence, since in the multiplication or division of any two quantities like signs produce always a positive and unlike signs a negative result, the product or quotient of any two geometrical magnitudes is to be regarded as positive or negative according as they have similar or opposite signs; and so, more generally, is the product of any number of magnitudes according as there happens to be an even or an odd number of negative signs amongst them.

The rectangle under any two lines being the same as their product, and the ratio of any two lines the same as their quotient: it follows from the above that the rectangle and the ratio

of any two lines have always the same sign, and are positive or negative together according as the lines themselves have similar or opposite signs. The square of every real line for the same reason is always positive, whether the line itself be positive or negative.

72. The terms "Arithmetic Mean" and "Geometric Mean," as employed in the geometry of the present day in reference to any number of magnitudes, ought for uniformity sake to bear the same relation to their "Arithmetic Sum" and "Geometric Sum" respectively. Such however is not the case, those terms having been employed to denote two entirely different things long before the consideration of signs had been forced on the attention of geometers, the former to denote *the n^{th} part of the sum*, and the latter to denote *the n^{th} root of the product* of any n magnitudes. In the same acceptations they are still employed, only with this difference, that in estimating the sum or product the signs as well as the absolute values of the several magnitudes are taken into account.

In geometry, therefore, the terms "Arithmetic Mean" and "Geometric Mean," in reference to any number of magnitudes, denote respectively *the n^{th} part of their geometric sum* and *the n^{th} root of their geometric product*, n being the number of the magnitudes. Hence, n being necessarily a positive integer, the former is positive or negative with the sum in every case, and the latter positive or negative with the product when n is odd, but real or imaginary and of either sign indifferently according as the product is positive or negative when n is even.

N.B. The term "Arithmetic Mean" is employed in geometry in the same sense as the term "mean" or "average" is employed in ordinary language.

73. Since by the evident *law of continuity*, as it is termed in geometry, a magnitude of any kind which varies *continuously* according to any law cannot possibly pass either in increase or decrease from any one value to any other *without passing through every intermediate value on the way*. It might appear at first sight as if a variable magnitude at the point of transition from positive to negative, or conversely, should necessarily pass *always* through the particular value 0. Such however

is not the case. Magnitudes susceptible of indefinite increase, as for instance the distance of a variable from a fixed point on a line, passing as often through ∞ as through 0 in changing sign.

To see this, if indeed it be not evident of itself from the example adduced, we have but to conceive two *reciprocal* magnitudes of any kind (8) to vary continuously, and either of them to change sign by passing through 0; for since the product of two such magnitudes is, from the nature of their connection, invariable both in magnitude and sign, every change of sign in either is necessarily accompanied by a simultaneous change of sign in the other, and every passage of either through 0 or ∞ by the simultaneous passage of the other through ∞ or 0, and therefore &c.

On the other hand, however, magnitudes unsusceptible of indefinite increase, and oscillating therefore as they vary between their extreme maxima and minima values (59), if they change sign at all, do so only by passing through 0 at each point of transition; thus for instance, the sine of an angle regarded as a magnitude, whose absolute value can never exceed 1 (60), changes sign only by passing through 0, its value whenever the angle itself in continuous increase or decrease $= \pm 2n$ right angles, n being any integer of the natural series 0, 1, 2, 3, 4, 5, 6, &c. to infinity.

74. In every application of the principle of signs, some method of notation which would indicate the directions, as well as represent the magnitudes, of the quantities considered would be of manifest convenience, and should as far as possible be systematically adhered to; the biliteral notation (4) which represents a magnitude of any kind by means of the two letters representing its extremities, whenever otherwise convenient, effects this purpose in as simple and expressive a manner as could be desired, by merely the order (4) in which the two letters are written.

Thus, a geometrical magnitude of any kind whose extremities are A and B is to be considered as measured, if represented by AB in the direction from A to B, and if by BA in the opposite direction from B to A. So that in accordance with the

convention of signs, AB is always to be regarded as $= -BA$, or which is the same thing $AB + BA = 0$, whatever be the nature of A and B and of the magnitude intercepted between them (3).

This premised, we proceed now to illustrate the convenience of the convention of signs by a few applications of very general utility in almost every department of pure and applied geometry.

75. *If A and B be any two points on a line, and P any third point taken arbitrarily on the same line, then whatever be the position of P with respect to A and B,*

$$AP - BP = AB,$$

regard being had to the signs as well as the magnitudes of the three intervals involved.

For, if 1°., P be external to AB at the side of B, then as AP, BP, and AB have all the same direction, and therefore the same sign, the relation is evident; if 2°. P be external to AB at the side of A, then, as by case 1°., $BP - AP = BA$, and as by the convention of signs $BA = -AB$, therefore &c. And if 3°., P be internal to AB, then as evidently $AP + PB = AB$, and as by the convention of signs $PB = -BP$, therefore &c.

A point P thus taken arbitrarily upon a line AB is said to divide that line, externally or internally according to its position, into two segments AP and BP, which, whether both measured from the extremities of the line to the point of section or from the point of section to the extremities of the line, have evidently similar or opposite directions, and therefore similar or opposite signs, according as the point of section is external or internal to the line. Hence the above relation expresses the general property that, *when a line AB is cut, externally or internally, at any point P, the geometrical difference* (70) *of the segments into which it is divided is constant and equal to the length of the line.*

The segments of a line AB divided at any point P having similar or opposite directions, and therefore similar or opposite signs, according as the point of section is external or internal to the line, *their rectangle and ratio are therefore both positive in the former case and negative in the latter.*

Hence, the problems "to divide a given line into segments, having a given rectangle or ratio," which would be ambiguous were the absolute magnitude of the rectangle or ratio alone given, becomes completely determinate when the sign also is given with it.

Cor. 1°. *If a line AB be cut, externally or internally, at any point P, then whatever be the position of P with respect to A and B,*

$$AP^2 + BP^2 = AB^2 + 2AP.BP;$$

regard being had to the signs as well as the magnitudes of the two segments AP and BP.

For, since by the above $AP - BP = AB$, whatever be the position of P, therefore $AP^2 + BP^2 - 2AP.BP = AB^2$, and therefore &c.

This relation being true for every position of P includes therefore the two properties (Euc. II. 7 and 4), the rectangle $AP.BP$ being positive or negative according as P is external or internal to AB.

Cor. 2°. *If from any point P a perpendicular PQ be let fall upon a line AB, then whatever be the position of P with respect to A and B,*

$$AP^2 - BP^2 = AB^2 + 2AB.BQ,$$

regard being had to the signs as well as the magnitudes of AB and BQ.

For by (Euc. I. 47, Cor.) $AP^2 - BP^2 = AQ^2 - BQ^2$, and by the preceding Cor. 1°., $AQ^2 = AB^2 + BQ^2 + 2AB.BQ$, therefore &c.

This relation being true for every position of P includes therefore the two properties (Euc. II. 12 and 13), the rectangle $AB.BQ$ being positive or negative according as the angle PBA is obtuse or acute.

76. *If A and B be any two points on a line, C the point on the line for which $AC + BC = 0$, and P any other point on the line, then whatever be the position of P,*

$$AP + BP = 2.CP \quad \ldots\ldots (1),$$

$$AP.BP = AC.BC + CP^2 \quad \ldots\ldots (2),$$

$$AP^2 + BP^2 = AC^2 + BC^2 + 2CP^2 \quad \ldots\ldots (3),$$

$$AP^2 - BP^2 = 2AB.CP \quad \ldots\ldots (4),$$

regard being had to the signs as well as the magnitudes of the several segments involved.

For, taking the sum, product, sum of squares, and difference of squares of the relations,

$$AP = AC + CP, \text{ and } BP = BC + CP,$$

which by the preceding (75) are true, whatever be the position of C, and remembering that by hypothesis $AC + BC = 0$, and that always $AC - BC = AB$, the above relations are the immediate results.

The point C on the line AB for which as above $AC + BC = 0$ being evidently the point of internal bisection of the line; the second of the above relations includes therefore the two properties (Euc. II. 5 and 6), and the third the two (Euc. II. 9 and 10), both being independent of the position of P; the first expresses that whatever be the position of P *the distance CP is the arithmetic mean of the distances AP and BP*; and the fourth, that whatever be the position of P *the difference of the squares of the distances AP and BP varies as the distance CP.* The four combined also supply obvious solutions of the four general problems: "To cut a line of given length, so that the sum, difference, sum of squares, or difference of squares of the segments, shall have a given magnitude and sign."

COR. *If AB and $A'B'$ be any two segments on the same line, C and C' their two middle points, then always*

$$CC' = \frac{AA' + BB'}{2} \text{ or } \frac{AB' + BA'}{2},$$

regard being had to the signs as well as the magnitudes of the several segments involved.

For, since for any arbitrary point P on the line, by the first of the above relations

$$2.CP = AP + BP \text{ and } 2.C'P = A'P + B'P,$$

therefore by subtraction

$$2(CP - C'P) = (AP - A'P) + (BP - B'P) \text{ or } (AP - B'P) + (BP - A'P),$$

and therefore as above (see 75)

$$2CC' = AA' + BB' \text{ or } AB' + BA'. \quad \text{Q.E.D.}$$

77. *If A and B be any two points on a line, a and b any two numbers positive or negative whose sum is not $=0$, O the point on the line for which $a.AO+b.BO=0$, and P any other point on the line, then, whatever be the position of P,*

$$a.AP+b.BP=(a+b).OP \text{..........................}(1),$$

$$a.AP^2+b.BP^2=a.AO^2+b.BO^2+(a+b).OP^2\text{...}(2),$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, since by (75), $AP=AO+OP$ and $BP=BO+OP$, whatever be the position of O, multiplying the first by a and the second by b and adding, then multiplying the square of the first by a and the square of the second by b and adding, remembering in both cases that by hypothesis $a.AO+b.BO=0$, the above relations are the immediate result.

Given the two points A and B and the two multiples a and b, to determine the point O, for which as above $a.AO+b.BO=0$, and which is evidently internal or external to AB according as a and b have similar or opposite signs. Assuming arbitrarily any point P on the line AB, and measuring from it a length PO equal in magnitude and sign to the sum $\dfrac{a.PA+b.PB}{a+b}$, the point O by the first of the above relations is that required, and by aid of it the two relations supply obvious solutions of the two following problems: "*on a given line AB to determine the point P for which the sum $a.AP+b.BP$ or the sum of the squares $a.AP^2+b.BP^2$ shall have a given magnitude and sign.*"

In the particular case when $a+b=0$, the value of PO, on which the position of O determined as above depends, being then infinite, the point O is therefore at an infinite distance, and the above relations both fail in consequence of their right-hand members becoming both indeterminate (13). Since, however, in that case $b=-a$, the sum

$$a.AP+b.BP=a.(AP-BP)=a.AB, \text{ (75)},$$

and therefore *is constant;* and the sum of the squares

$$a.AP^2+b.BP^2=a\,(AP^2-BP^2)=2a.AB.CP, \text{ (76)},$$

C being the middle point of AB, and therefore *varies as CP;*

relations simpler than those for the general case where $a+b$ is not $=0$.

COR. *If from the three points A, B, and O, perpendiculars or any other isoclinals AL, BL, and OL be let fall upon any arbitrary line L, then, whatever be the position of L,*

$$a.AL+b.BL=(a+b).OL,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, in the particular case when L is parallel to AB, since then $AL=BL=OL$ the relation is evident; and in any other case if P be the point in which L intersects AB, since by similar triangles $AL:BL:OL=AP:BP:OP$, and since by the first of the above relations $a.AP+b.BP=(a+b)\,OP$, therefore &c.

78. *If A, B, C, D, E, F, &c. be any number of points on a line, situated in any manner with respect to each other, then, whatever be their order and disposition—*

For every three of them A, B, C,

$$AB+BC+CA=0.$$

For every four of them A, B, C, D,

$$AB+BC+CD+DA=0.$$

For every five of them A, B, C, D, E,

$$AB+BC+CD+DE+EA=0;$$

and so on for any number, the last being always connected with the first in completing the circuit, and the signs as well as the magnitudes of the several intercepted segments being always taken in account in the summation.

For, since by (75),

$$AB+BC=AC,\quad AC+CD=AD,$$
$$AD+DE=AE,\quad AE+EF=AF,\ \text{\&c.},$$

therefore, $$AB+BC+CA=AC+CA=0,$$
$$AB+BC+CD+DA=AD+DA=0,$$
$$AB+BC+CD+DE+EA=AE+EA=0,\ \text{\&c.},$$

and therefore &c. Q.E.D.

79. *If A, B, C, D, &c. be any number (n) of points on a line, disposed in any manner, O the point on the line for which* $AO + BO + CO + DO + \&c. = 0$, *and P any other point on the line, then, whatever be the position of P,*

$$AP + BP + CP + DP + \&c. = n.OP \ldots\ldots\ldots\ldots (1),$$

$$AP^2 + BP^2 + CP^2 + DP^2 + \&c. = AO^2 + BO^2 + CO^2 + DO^2 + \&c. + n.OP^2 \ldots\ldots\ldots\ldots (2),$$

the signs as well as the magnitudes of the several segments being taken into account in the first.

For, taking the sum and the sum of the squares of the several relations $AP = AO + OP$, $BP = BO + OP$, $CP = CO + OP$, $DP = DO + OP$, &c., which by (75) are true whatever be the position of O, and remembering that, by hypothesis,

$$AO + BO + CO + DO + \&c. = 0,$$

the above relations are the immediate result.

The point O on the line for which, as above,

$$AO + BO + CO + DO + \&c. = 0,$$

or, as it may be more concisely written, $\Sigma(AO) = 0$, being such by relation 1, that for every other point P on the line *the distance OP is the arithmetic mean of the several distances AP, BP, CP, DP, &c.*, is termed, in consequence, *the mean centre of the system of points A, B, C, D, &c.;* and to determine its position when the latter are given, we have but to assume arbitrarily any point P on the line, and to measure from it a distance PO equal in magnitude and sign to the n^{th} part of the sum of the distances PA, PB, PC, PD, &c., or, as it may be more concisely written, $= \dfrac{\Sigma(PA)}{n}$; the point O, by relation 1, is that required, and by its aid the two relations 1 and 2 supply obvious solutions of the two general problems: "*Given any number of points A, B, C, D, &c. on a line, to determine the point P on the line for which the sum* $\Sigma(AP)$ *or the sum of the squares* $\Sigma(AP^2)$ *shall be given.*"

Cor. 1°. *If at the mean centre O a perpendicular OS be erected to the line whose square* $OS^2 =$ *the* n^{th} *part of the sum of the squares* $\Sigma(AO^2)$, *then for any point P on the line the sum of the squares* $\Sigma(AP^2) = n.SP^2$.

For, since by relation 2, $\Sigma(AP^2) = \Sigma(AO^2) + n.OP^2$, and since by construction, $\Sigma(AO^2) = n.OS^2$, therefore

$$\Sigma(AP^2) = n(OS^2 + OP^2) = n.SP^2.$$

Hence the variable sum $\Sigma(AP^2)$ has equal values for every two points on the line equidistant from O, and the minimum value for the point O itself.

Cor. 2°. Since when, as in relation 1, $\Sigma(AP) = n.OP$, then, as in relation 2,

$$\Sigma(AP^2) = n.OP^2 + \Sigma(AO^2) = n.OP^2 + \Sigma(AP - OP)^2,$$

it follows consequently that—

When the same sum, $\Sigma(AP)$, is cut into any number n of unequal parts AP, BP, CP, DP, &c., and also into the same number n of equal parts OP, OP, OP, OP, &c., the sum of the squares of the n unequal parts $\Sigma(AP^2)$ is equal to the sum of the squares of the n equal parts $n.OP^2$ + the sum of the squares of the n differences $\Sigma(AP - OP)^2$.

Cor. 3°. *If A, B, C, D, &c. and A', B', C', D', &c. be two systems of any common number of points on the same line, O and O' their mean centres, and n their common number of points, then*

$$OO' = \frac{AA' + BB' + CC' + DD' + \&c.}{n}$$

any mode of correspondence between the points of the systems in pairs being adopted in the summation.

For since, for any arbitrary point P on the line, by relation 1,

$$n.OP = AP + BP + CP + DP + \&c.,$$

and $$n.O'P = A'P + B'P + C'P + D'P + \&c.$$

therefore

$$n.(OP - O'P) = (AP - A'P) + (BP - B'P)$$
$$+ (CP - C'P) + (DP - D'P) + \&c.,$$

or (75) $$n.OO' = AA' + BB' + CC' + DD' + \&c.,$$

and therefore &c.

Cor. 4°. *If A, B, C, D, &c. and A', B', C', D', &c. be any two systems of points on the same line, O and O' their mean centres, and n and n' their numbers of points, then*

$$OO' = \frac{\Sigma(AA')}{nn'},$$

every point of one system being combined in the summation with every point of the other.

For, adding together the several relations,

$$AA' + BA' + CA' + DA' + \&c. = n.OA',$$
$$AB' + BB' + CB' + DB' + \&c. = n.OB',$$
$$AC' + BC' + CC' + DC' + \&c. = n.OC',$$
$$AD' + BD' + CD' + DD' + \&c. = n.OD', \&c.$$

there results at once the relation

$$\Sigma(AA') = n.(OA' + OB' + OC' + OD' + \&c.) = n.\Sigma(OA') = nn'.OO',$$

and therefore &c.

80. *If A, B, C, D, &c. be any system of points on a line, disposed in any manner, a, b, c, d, &c. any system of corresponding multiples, positive or negative, whose sum is not $= 0$, O the point on the line for which*

$$a.AO + b.BO + c.CO + d.DO + \&c. = 0,$$

and P any other point on the line, then, whatever be the position of P,

$$a.AP + b.BP + c.CP + d.DP + \&c. = (a+b+c+d+\&c.).OP \ldots (1),$$

$$a.AP^2 + b.BP^2 + c.CP^2 + d.DP^2 + \&c.$$
$$= a.AO^2 + b.BO^2 + c.CO^2 + d.DO^2 + \&c.$$
$$+ (a+b+c+d+\&c.).OP^2 \ldots\ldots\ldots\ldots (2),$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, multiplying the several relations $AP = AO + OP$, $BP = BO + OP$, $CP = CO + OP$, $DP = DO + OP$, &c., which, by (75), are true whatever be the position of O, and also their squares, by the several corresponding multiples a, b, c, d, &c., and adding, remembering in both cases that by hypothesis $\Sigma(a.AO) = 0$, the above relations are the immediate result.

The point O on the line, for which as above $\Sigma(a.AO) = 0$, is termed, in virtue of relation 1, *the mean centre of the system of points A, B, C, D, &c. for the system of multiples a, b, c, d, &c.;* and to determine its position when the several points and multiples are given, we have but to assume arbitrarily any point P on the line, and to measure from it a length PO equal

in magnitude and sign to $\frac{\Sigma(a.PA)}{\Sigma(a)}$, the point O, by relation 1, is that required, and by its aid the two relations 1 and 2 supply obvious solutions of the two general problems: "*Given any number of points on a line A, B, C, D, &c., and the same number of corresponding multiples a, b, c, d, &c. whose sum is not $=0$. To determine the point P on the line for which the sum $\Sigma(a.AP)$, or the sum of the squares $\Sigma(a.AP^2)$ shall be given.*"

In the particular case when $\Sigma(a)=0$, the value of PO, as given by the above formula, being then infinite, the point O is therefore at an infinite distance, and the relations 1 and 2 both fail in consequence of their right-hand members becoming both indeterminate (13). This case, the laws of which, though simpler, differ altogether from those of the general case when $\Sigma(a)$ is not $=0$, will be considered separately in the next section.

COR. 1°. *If round the mean centre O as centre and with a radius OP whose square equal to the absolute value of* $\frac{\Sigma(a.AO^2)}{\Sigma(a)}$, *disregarding its sign, a circle be described intersecting the line at the points M and N, and the perpendicular to it through O in either direction at the point S, then for any point P on the line the sum of squares $\Sigma(a.AP^2)=\Sigma(a).SP^2$ or $\Sigma(a).MP.NP$, according as $\Sigma(a)$ and $\Sigma(a.AO^2)$ have similar or opposite signs.*

For since, by relation 2, $\Sigma(a.AP^2)=\Sigma(a).OP^2+\Sigma(a.AO^2)$, and since by construction $\Sigma(a.AO^2)=\pm\Sigma(a).OR^2$, therefore

$$\Sigma(a.AP^2)=\Sigma(a).(OP^2\pm OR^2)=\Sigma(a).SP^2 \text{ or } \Sigma(a).MP.NP.$$

Hence, in both cases, the variable sum $\Sigma(a.AP^2)$ has equal values for every two points on the line equidistant from O, and the minimum value for the point O itself; it being remembered however that as it vanishes in the second case at the two points M and N, and increases negatively from each up to O, the term *minimum* is to be understood in the sense of *negative maximum* in that case, see (58).

COR. 2°. *If a system of any number of points on a line A, B, C, D, &c., and their mean centre O for any system of multiples a, b, c, d, &c., be projected by perpendiculars or any other parallels AA', BB', CC', DD', &c., and OO' upon any arbitrary line L, then, whatever be the position of L.*

a. The projection O' of the mean centre is the mean centre of the projections A', B', C', D', &c. of the points for the same system of multiples.

b. The projector OO' of the mean centre, is the mean of the projectors AA', BB', CC', DD', &c. of the points for the same system of multiples.

For, as in Cor. 1 (77), for the case of two points. If L be parallel to the line of the points, both properties are evident; and in any other position, if P be the point in which the two lines intersect, since by similar triangles,

$$AP : BP : CP : DP, \&c. : OP = A'P : B'P : C'P : D'P, \&c.: O'P$$
$$= AA' : BB' : CC' : DD', \&c. : OO',$$

and since by relation 1,

$$\Sigma\,(a.AP) = \Sigma\,(a).\,OP; \text{ therefore } \Sigma\,(a.A'P) = \Sigma\,(a).\,O'P,$$

and $\Sigma\,(a.AA') = \Sigma\,(a).\,OO'$, and therefore &c.

Cor. 3°. *If A, B, C, D, &c. and A', B', C', D', &c. be two systems of any common number of points on the same line, O and O' their mean centres for any common system of multiples a, b, c, d, &c. then*

$$OO' = \frac{a.AA' + b.BB' + c.CC' + d.DD' + \&c.}{a + b + c + d + \&c.},$$

pairs of points having common multiples being combined in the summation.

For, since for any arbitrary point P on the line, by relation 1,

$$\Sigma\,(a).OP = a.AP + b.BP + c.CP + d.DP + \&c.,$$

and $\Sigma\,(a).O'P = a.A'P + b.B'P + c.C'P + d.D'P + \&c.$,

therefore

$$\Sigma\,(a).(OP - O'P) = a.(AP - A'P) + b.(BP - B'P)$$
$$+ c.(CP - C'P) + a.(DP - D'P) + \&c.$$

and therefore as above

$$\Sigma\,(a).\,OO' = a.AA' + b.BB' + c.CC' + d.DD' + \&c.$$

Cor. 4°. *If A, B, C, D, &c. and A', B', C', D', &c. be any two systems of points on the same line, O and O' their mean centres for any two systems of multiples a, b, c, d, &c., and a', b', c', d', &c., then*

$$OO' = \frac{\Sigma\,(aa'.AA')}{\Sigma\,(a).\Sigma\,(a')},$$

every point of one system being combined in the summation with every point of the other.

For, adding together the several relations

$$a.AA' + b.BA' + c.CA' + d.DA' + \&c. = \Sigma(a).OA',$$
$$a.AB' + b.BB' + c.CB' + d.DB' + \&c. = \Sigma(a).OB',$$
$$a.AC' + b.BC' + c.CC' + d.DC' + \&c. = \Sigma(a).OC',$$
$$a.AD' + b.BD' + c.CD' + d.DD' + \&c. = \Sigma(a).OD', \&c.$$

multiplied respectively by a', b', c', d', &c. there results immediately the relation

$$\Sigma(aa'.AA') = \Sigma(a).\Sigma(a'.OA') = \Sigma(a).\Sigma(a').OO',$$

and therefore &c.

81. *If A, B, C, D, &c. be any system of points on a line disposed in any manner, a, b, c, d, &c. any system of corresponding multiples, some positive and some negative, whose sum $= 0$, then for every point P on the line not at infinity the sum $\Sigma(a.AP)$ has the same constant value.*

In the same case, if I be the point on the line for which the sum $\Sigma(a.AI^2) = 0$, then for every other point P on the line the sum $\Sigma(a.AP^2) = 2k.IP$, k being the constant value of the sum $\Sigma(a.AP)$ for every point on the line.

To prove the first,—since for any two points P and Q on the line by (75),

$$AP - AQ = QP,\ BP - BQ = QP,\ CP - CQ = QP,\ DP - DQ = QP, \&c.$$

therefore, multiplying by a, b, c, d, &c. and adding,

$$\Sigma(a.AP) - \Sigma(a.AQ) = \Sigma(a).QP = 0,$$

when $\Sigma(a) = 0$, whatever be the positions of P and Q, provided neither of them be at infinity, and therefore &c.

To prove the second,—since for any two points P and Q on the line by (76 (4)),

$$AP^2 - AQ^2 = 2AR.QP, \quad BP^2 - BQ^2 = 2BR.QP,$$
$$CP^2 - CQ^2 = 2CR.PQ, \quad DP^2 - DQ^2 = 2DR.PQ, \&c.$$

R being the middle point of PQ; therefore, multiplying by a, b, c, d, &c. and adding,

$$\Sigma(a.AP^2) - \Sigma(a.AQ^2) = 2\Sigma(a.AR).QP = 2k.QP,$$

when $\Sigma(a) = 0$, whatever be the positions of P and Q; and therefore when either of them Q is the particular point I for which $\Sigma(a.AI^2) = 0$, then for the other P, whatever be its position, $\Sigma(a.AP^2) = 2k.IP$, as above stated.

From the above relations it appears that, while the sum $\Sigma(a.AP)$ is invariable, the sum $\Sigma(a.AP^2)$ follows a very simple law of variation when $\Sigma(a) = 0$, being simply proportional to the distance of P from a certain point I on the line, admitting therefore of no maximum or minimum value, but susceptible of every value positive and negative from 0 to ∞, passing through infinity as P passes through infinity, and through nothing as P passes through I, and changing from positive to negative, and conversely, at the passage through each.

To determine the point I, when the several points A, B, C, D, &c. and the several multiples a, b, c, d, &c. are given; assuming arbitrarily any point P on the line, and measuring from it a length PI equal in magnitude and sign to

$$\frac{\Sigma(a.PA^2)}{-2k} = \frac{\Sigma(a.PA^2)}{2.\Sigma(a.PA)},$$

the point I, by relation 2, is that required, and by its aid the same relation supplies an obvious solution of the more general problem, "*to determine the point P on the line for which the $\Sigma(a.PA^2)$ shall have any given magnitude and sign.*"

In the particular case where the constant $k = 0$, the value of PI, as given by the above formula, being then infinite, the point I is therefore at an infinite distance, and the relation $\Sigma(a.AP^2) = 2k.IP$ fails in consequence of its right-hand member becoming indeterminate (13). In that case however it is easy to see that, as it ought, *the sum $\Sigma(a.AP^2)$ has the same constant value for every point on the line not at infinity.*

For since for every two points P and Q on the line, as above shown, $\Sigma(a.AP^2) - \Sigma(a.AQ^2) = 2k.QP$, whatever be the value of k, therefore when $k = 0$, $\Sigma(a.AP^2) = \Sigma(a.AQ^2)$, whatever be the positions of P and Q, provided neither of them be at infinity, and therefore &c.

Among the various ways in which the constant k may be represented in the form of a single quantity, when the several points A, B, C, D, &c. and the several multiples a, b, c, d, &c. are given, the following is perhaps the most convenient.

Conceiving the entire system of points $\Sigma(A)$ *divided into two distinct groups, one* $\Sigma(A_+)$ *corresponding to the positive, and the other* $\Sigma(A_-)$ *to the negative multiples. If* O_+ *and* O_- *be the mean centres of the two groups for their respective systems of multiples* $\Sigma(a_+)$ *and* $\Sigma(a_-)$, *the constant sum*

$$\Sigma(a.AP) = \Sigma(a_+).O_+O_-, \text{ or } = \Sigma(a_-).O_-O_+.$$

For, $\Sigma(a.AP) = \Sigma(a_+.A_+P) + \Sigma(a_-.A_-P) = \Sigma(a_+).O_+P$
$+ \Sigma(a_-).O_-P$, by (80),

but $\Sigma(a_+) + \Sigma(a_-) = \Sigma(a) = 0$, by hypothesis,

therefore $\Sigma(a.AP) = \Sigma(a_+).(O_+P - O_-P)$,

or $= \Sigma(a_-).(O_-P - O_+P) = \Sigma(a_+).O_+O_-$, or $= \Sigma(a_-).O_-O_+$.

Hence when the two points O_+ *and* O_- *coincide, the constant* $k = 0$ *at all points of the line.*

In every case where the constant $k = 0$, *the position of the mean centre* O *of the entire system of points* $\Sigma(A)$ *for the entire system of multiples* $\Sigma(a)$ *is indeterminate.* The relation $\Sigma(a.AO) = 0$, by which that point, in general unique, is characterized (80), being then satisfied indifferently by *every* point on the line. An example of this for the case of three points will be given in the next number.

Hence, generally, the position of the mean centre O of any system of points A, B, C, D, &c. on a line for any system of multiples a, b, c, d, &c. whose sum $= 0$, is either indeterminate or impossible *at any finite distance*, indeterminate if the value of the constant $k = 0$, impossible if not.

82. *If* A, B, C, D *be four points on a line disposed in any manner, then always, none of the four being at infinity,*

$$BC.AD + CA.BD + AB.CD = 0,$$

regard being had to the signs as well as the magnitudes of the six segments involved.

For since whatever be the positions of the four points (75),

$$AD - CD = AC, \text{ and } BD - CD = BC,$$

therefore, multiplying the first by BC and the second by AC, and subtracting

$$BC.AD + CA.BD + (AC - BC)\,CD = 0,$$

the same as above, $AC - BC$ being always $= AB$ (75).

Hence, (see preceding article), *the mean centre O of three points A, B, C on a line for three multiples a, b, c, proportional in magnitude and sign to the three intervals BC, CA, AB is indeterminate.* Every point P on the line in virtue of the above relation, satisfying indifferently the characteristic condition,

$$a.AP + b.BP + c.CP = 0.$$

As four points on a line A, B, C, D, however disposed, determine in every case six different segments corresponding to each other two and two in three different sets of opposite pairs BC and AD, CA and BD, AB and CD, the above is the general relation connecting those six segments in all cases, regard being had to their signs as well as their magnitudes, and interpreted absolutely, disregarding signs, it expresses evidently the general property that—

Whatever be the disposition of four points on a line the rectangle under one pair of opposites of the six segments they determine is numerically equal to the sum of the rectangles under the other two pairs.

If the four points in the order of their disposition be denoted by 1, 2, 3, 4 respectively, it is easy to see that in all cases the rectangle $\overline{13}.\overline{24}$ is the one that is equal to the sum of the other two $\overline{12}.\overline{34}$ and $\overline{23}.\overline{14}$; for denoting by x, y, z the absolute intervals from 1 to 2, 2 to 3, 3 to 4, respectively, disregarding their signs, the relation

$$(x+y)(y+z) = xz + y(x+y+z),$$

is evidently in all cases identically true.

COR. 1°. *If A, B, C, D be four points on a line disposed in any manner, and O any point whatever not at infinity, then always*

$$\textit{area } BOC.\textit{area } AOD + \textit{area } COA.\textit{area } BOD + \textit{area } AOB.\textit{area } COD = 0,$$

regard being had to their signs as well as their magnitudes.

For in the relation $BC.AD + CA.BD + AB.CD = 0$, multiplying each segment by half the length of the perpendicular from O on the line, the relation just given is the immediate result.

COR. 2°. *More generally if A, B, C, D be any four points*

and O any fifth point, none of the five being at infinity, then always

$$\text{area } BOC \,.\, \text{area } AOD + \text{area } COA \,.\, \text{area } BOD + \text{area } AOB \,.\, \text{area } COD = 0,$$

regard being had to their signs as well as their magnitudes.

For conceiving the four lines AO, BO, CO, DO, met by any fifth line L not parallel to one of themselves in the four points A', B', C', D', since then (64)

$$\text{area } BOC : \text{area } B'OC' = OB \,.\, OC : OB' \,.\, OC',$$

and $$\text{area } AOD : \text{area } A'OD' = OA \,.\, OD : OA' \,.\, OD',$$

both pairs of triangles having the same angles at O; therefore

$$\text{area } BOC \,.\, \text{area } AOD : \text{area } B'OC' \,.\, \text{area } A'OD' = OA \,.\, OB \,.\, OC \,.\, OD : OA' \,.\, OB' \,.\, OC' \,.\, OD',$$

and (both remaining pairs of corresponding products having for the same reason the same ratio) therefore

$$\text{area } BOC.\text{area } AOD : \text{area } COA.\text{area } BOD : \text{area } AOB.\text{area } COD$$
$$= \text{area } B'OC' \,.\, \text{area } A'OD' : \text{area } C'OA' \,.\, \text{area } B'OD' : \text{area } A'OB' \,.\, \text{area } C'OD';$$

but by Cor. 1°. the sum of the three consequents $= 0$, therefore, also the sum of the three antecedents $= 0$, and therefore &c.

COR. 3°. *If OA, OB, OC, OD be four lines passing through a point, then in all cases whatever be their directions,*

$$\sin BOC.\sin AOD + \sin COA \,.\, \sin BOD + \sin AOB.\sin COD = 0,$$

regard being had to the signs as well as the magnitudes of the six angles involved.

For, if A, B, C, D be the four points in which any line not passing through O intersects the four lines; since then by (64)

$OB.OC.\sin BOC = 2 \text{area} BOC$ and $OA.OD.\sin AOD = 2 \text{area} AOD$, therefore

$$OA \,.\, OB.OC.OD.\sin BOC.\sin AOD = 4 \text{ area} .BOC \,.\, \text{area } AOD,$$

and, similar relations for the same reason existing for the other two pairs, therefore,

$$\sin BOC \,.\, \sin AOD : \sin COA \,.\, \sin BOD : \sin AOB \,.\, \sin COD$$
$$= \text{area} BOC.\text{area} AOD : \text{area} COA.\text{area} BOD : \text{area} AOB.\text{area} COD,$$

but by Cor. 1°. the sum of the three antecedents $=0$, therefore also the sum of the three consequents $=0$, and therefore &c.

Otherwise thus, if A, B, C, D, be the four points in which any circle passing through O intersects the four lines, then since (62) diameter of circle . $\sin BOC = \text{chord}\, BC$, and diameter of circle . $\sin AOD = \text{chord}\, AD$; therefore diameter² of circle . $\sin BOC . \sin AOD = \text{chord}\, BC . \text{chord}\, AD$, and (similar relations for the same reason existing for the other two pairs) therefore

$$\sin BOC . \sin AOD : \sin COA . \sin BOD : \sin AOB . \sin COD$$
$$= \text{chord}\, BC . \text{chord}\, AD : \text{chord}\, CA . \text{chord}\, BD : \text{chord}\, AB . \text{chord}\, CD;$$

but by Ptolemy's theorem (Euc. VI. 16, Cor.) one of the three consequents is always numerically equal to the sum of the other two, therefore, disregarding signs, the same is true also of the three antecedents, and therefore &c.

Cor. 4°. *If A, B, C be any three points in a line, and AL, BL, CL their three distances perpendicular or in any common direction from any line L not at infinity, then always*

$$BC . AL + CA . BL + AB . CL = 0,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, if L be parallel to the line containing the points, then since $AL = BL = CL$, and since by (78) $BC + CA + AB = 0$, therefore &c., and if not, then if P be the intersection of the two lines, since $AL : BL : CL :: AP : BP : CP$, and since by the above $BC . AP + CA . BP + AB . CP = 0$, therefore &c.

Cor. 5°. *If L, M, N be any three parallel lines and PL, PM, PN their three distances perpendicular or in any common direction from any point P not at infinity, then always*

$$MN . PL + NL . PM + LM . PN = 0,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For if A, B, C be the three points in which any line through P not parallel to their common direction intersects L, M, N, then since $MN : NL : LM : PL : PM : PN :: BC : CA : AB : PA : PB : PC$, and since by the above $BC . PA + CA . PB + AB . PC = 0$, therefore &c.

Cor. 6°. *If L, M, N be any three lines passing through a point O, and PL, PM, PN the three perpendiculars or any other isoclinals upon them from any point P not at infinity, then always*

$$\sin MN\,.\,PL + \sin NL\,.\,PM + \sin LM\,.\,PN = 0,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, dividing by PO the distance of P from O, or more generally by the diameter of the circle passing through P and O and through the feet of the three perpendiculars or isoclinals, the relation becomes evidently identical with that of Cor. 3°. for the four lines OL, OM, ON, OP, and therefore &c.

The three sides of every triangle being as the three sines of the opposite angles (63), the three sines in the preceding formula may therefore be replaced by the three sides of any triangle formed by parallels to the three lines.

Cor. 7°. *If α, β, γ be the three angles of any triangle, and α', β', γ' those at which the three opposite sides a, b, c intersect any line d, then always*

$$\sin\alpha\,.\,\sin\alpha'\,. + \sin\beta\,.\,\sin\beta' + \sin\gamma\,.\,\sin\gamma' = 0,$$

regard being had to the signs as well as the magnitudes of the six angles involved.

For, drawing through any arbitrary point O four lines OA, OB, OC, OD parallel to a, b, c, d, then since by parallels $BOC=\alpha$, $COA=\beta$, $AOB=\gamma$, and $AOD=\alpha'$, $BOD=\beta'$, $COD=\gamma'$, the relation is evident from that of Cor. 3°.

83. *If A, B, C be three points on a line disposed in any manner, and AP, BP, CP the three lines connecting them with any point P not at infinity, then always*

$$BC.AP^2 + CA.BP^2 + AB.CP^2 = -\,BC.CA.AB,$$

regard being had to the signs as well as the magnitudes of the three segments involved.

For letting fall from P the perpendicular PQ on the line, then since (75 Cor. 2°.)

$$AP^2 - CP^2 = AC^2 + 2\,AC.CQ,$$

and $$BP^2 - CP^2 = BC^2 + 2\,BC.CQ,$$

therefore, multiplying the first by BC and the second by AC, and subtracting,

$$BC.AP^2 + CA.BP^2 + (AC - BC).CP^2 = BC.AC^2 - AC.BC^2$$
$$= - BC.CA.(AC - BC)$$

the same as the above, $AC - BC$ being always $= AB$ (75).

From the above which is *the general relation connecting any three lines drawn from a point to a line, and the three segments they intercept on the line;* it is evident that *when A, B, and C are fixed, the sum $BC.AP^2 + CA.BP^2 + AB.CP^2$ is independent of the position of P and therefore constant for all points at a finite distance;* an example of the general property established in (81), that when, as in the present instance (see preceding article), the sum $\Sigma\,(a\,.\,AQ)$ is *nothing* for every point on a line, then the sum $\Sigma\,(a.AQ^2)$ is *constant* for every point on the line, and therefore for every point whatever not at infinity, the quantity $\Sigma\,(a)\,PQ^2$ by which the sums for the two points P and Q differ, Euc. I. 47, vanishing with $\Sigma\,(a)$ for every position of P for which PQ is not infinite.

Dividing both sides of the above relation by its right-hand member $- BC.CA.AB$, it assumes the not less symmetrical but more compact form

$$\frac{AP^2}{AB.AC} + \frac{BP^2}{BC.BA} + \frac{CP^2}{CA.CB} = 1,$$

regard being had of course to the signs as well as the magnitudes of the three rectangles $AB.AC$, $BC.BA$, $CA.CB$ in the addition.

Cor. 1°. *If A, B, C be three points on a line disposed in any manner, and AR, BS, CT the three tangents from them to any circle, not either at infinity or infinite in radius, then always*

$$BC.AR^2 + CA.BS^2 + AB.CT^2 = - BC.CA.AB,$$

regard being had to the signs as well as the magnitudes of all the quantities involved.

For, if P be the centre of the circle, then since

$$AR^2 = AP^2 - PR^2, \quad BS^2 = BP^2 - PS^2, \quad CT^2 = CP^2 - PT^2,$$

and since $PR = PS = PT =$ radius of circle, therefore

$$BC.AR^2 + CA.BS^2 + AB.CT^2 = BC.AP^2 + CA.BP^2$$
$$+ AB.CP^2 - (BC + CA + AB).\text{radius}^2 \text{ of circle},$$

the first part of which by the above $= - BC.CA.AB$, and the second point of which by (78) $= 0$, and therefore &c.

Dividing, as in the original, both sides of this latter relation by its right-hand member $-BC.CA.AB$, it too assumes the more compact and not less symmetrical form

$$\frac{AR^2}{AB.AC}+\frac{BS^2}{BC.BA}+\frac{CT^2}{CA.CB}=1,$$

regard again of course being had to the signs as well as the magnitudes of all the quantities involved.

COR. 2°. *If CZ be any line drawn from the vertex C to the base AB of any triangle ACB, then always*

$$AZ.CB^2-BZ.CA^2=AB.(CZ^2-AZ.BZ),$$

regard being had to the signs as well as the magnitudes of the three intercepts AZ, BZ, and AB.

This relation is obviously the same as the above, only stated in the form in which it most naturally presents itself in the process by which it was established above.

The following particular cases are deserving of notice:

1°. If Z bisect AB, then $AZ=\frac{1}{2}AB$ and $BZ=-\frac{1}{2}AB$, and the relation becomes

$$CZ^2-AZ.BZ=\tfrac{1}{2}(CA^2+CB^2),$$

the known relation connecting the base, bisector of base, and sides of a triangle, (Euc. II. 12, 13, Cor.).

2°. If CZ bisect ACB externally or internally, then as $AZ:BZ=\pm AC:BC$, (Euc. VI. 3), therefore $AZ.CB=\pm BZ.CA$ according as the bisection is external or internal, and the relation, remembering that in either case $AZ-BZ=AB$ (75), becomes

$$CZ^2-AZ.BZ=\mp CA.CB,$$

the known relation connecting the sides of a triangle, either bisector external or internal of the vertical angle, and the segments into which it divides the base.

3°. If the triangle be isosceles, then $CA=CB$, and the relation, remembering as before that always $AZ-BZ=AB$, becomes $CZ^2-AZ.BZ=CA^2$ or CB^2 or $CA.CB$,

the known relation connecting either side of an isosceles triangle, any line drawn from the vertex to the base, and the rectangle under the segments into which it divides the base (Euc. II. 5, 6, Cor.).

4°. If the triangle be right-angled, then $CA^2 + CB^2 = AB^2$, and the relation, multiplying its two sides, the first by $AZ - BZ$, and the second by its equivalent AB, which causes the rectangle $AZ.BZ$ to disappear in virtue of the property of the triangle, becomes

$$BC^2.AZ^2 + AC^2.BZ^2 = AB^2.CZ^2,$$

the general relation connecting the sides and the distances of any point on the hypotenuse from the vertices of a right-angled triangle.

COR. 3°. *If A, B, C, D be any four points on a circle taken in the order of their disposition, and P any fifth point without, within, or upon the circle, but not at infinity, then always*

$$\textit{area}\, BCD.AP^2 - \textit{area}\, CDA.BP^2 + \textit{area}\, DAB.CP^2 - \textit{area}\, ABC.DP^2 = 0,$$

regard being had only to the absolute magnitudes of the several areas which from their disposition are incapable of being compared in sign.

For, joining P with the intersection O of the two chords AC and BD, which from their positions necessarily intersect internally; then from the relation, Cor. 1°., applied successively to the two triangles APC and BPD, disregarding all signs in each, and attending only to absolute values throughout,

$$CO.AP^2 + AO.CP^2 = AC.(PO^2 + AO.CO),$$

$$DO.CP^2 + BO.DP^2 = BD.(PO^2 + BO.DO),$$

from which, as $AO.CO = BO.DO$, (Euc. III. 35), therefore immediately

$$BD.CO.AP^2 + BD.AO.CP^2 = AC.DO.BP^2 + AC.BO.DP^2,$$

which is evidently identical with the other, the four rectangles $BD.CO$, &c. multiplied each by the sine of the angle of intersection of the two chords AC and BD being respectively the double areas of the four triangles BCD, &c.

This theorem is due to Dr. Salmon, who has given it in his *Conic Sections* as the geometrical interpretation of the analytical condition that four points A, B, C, D should lie on a circle.

COR. 4°. *If A, B, C, D be any four points on a circle taken*

in the order of their disposition, and AQ, BR, CS, DT the four tangents from them to another circle not either infinitely distant or infinite in radius, then always

$$\textit{area}\, BCD.AQ^2 - \textit{area}\, CDA.BR^2 + \textit{area}\, DAB.CS^2 - \textit{area}\, ABC.DT^2 = 0,$$

regard being had, as in Cor. 3°., only to the absolute values of the several areas.

For, if P be the centre of the latter circle, then since

$$PQ = PR = PS = PT = \text{radius of that circle},$$

and since

$$\text{area}\, BCD + \text{area}\, DAB = \text{area}\, CDA + \text{area}\, ABC = \text{area of quadrilateral}\, ABCD,$$

therefore

$$\text{area}\, BCD.PQ^2 - \text{area}\, CDA.PR^2 + \text{area}\, DAB.PS^2 - \text{area}\, ABC.PT^2 = 0,$$

which relation, subtracted from that of Cor. 3°., leaves immediately that just stated, and therefore &c.

If in this relation, as in that of Cor. 1°., any of the points A, B, C, D be within the second circle, the squares of the corresponding tangents are of course negative.

Cor. 5°. *If OA, OB, OC, OD be four lines passing through a point, then in all cases, whatever be their directions,*

$$\frac{\sin BOD.\sin COD}{\sin BOA.\sin COA} + \frac{\sin COD.\sin AOD}{\sin COB.\sin AOB} + \frac{\sin AOD.\sin BOD}{\sin AOC.\sin BOC} = 1,$$

regard being had to the signs as well as the magnitudes of the six angles involved.

For, drawing any line L parallel to OD, meeting OA, OB, OC in A, B, C, then since (63)

$$\frac{\sin BOD}{\sin BOA} = \frac{AO}{AB} \text{ and } \frac{\sin COD}{\sin COA} = \frac{AO}{AC};$$

therefore
$$\frac{\sin BOD.\sin COD}{\sin BOA.\sin COA} = \frac{AO^2}{AB.AC},$$

and, similarly,

$$\frac{\sin COD.\sin AOD}{\sin COB.\sin AOB} = \frac{BO^2}{BC.BA} \text{ and } \frac{\sin AOD.\sin BOD}{\sin AOC.\sin BOC} = \frac{CO^2}{CA.CA};$$

but, by the original relation of the present article, the sum of the three right-hand members $= 1$, therefore also the sum of the left-hand members $= 1$, and therefore &c.

Cor. 6°. *If OA, OB, OC be three lines passing through a point, and PA, PB, PC the three perpendiculars upon them from any point P not at infinity, then always whatever be their directions*

$$PB.PC.\sin BOC + PC.PA.\sin COA + PA.PB.\sin AOB$$
$$= -PO^2.\sin BOC.\sin COA.\sin AOB,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, dividing both sides of the relation by its right-hand member $-PO^2.\sin BOC.\sin COA.\sin AOB$, the relation of Cor. 5°., for the four lines OA, OB, OC, and OP, is the immediate result, and therefore &c.

Cor. 7°. *If α, β, γ be the three angles of any triangle, and α', β', γ' those at which the three opposite sides a, b, c intersect any line d, then always*

$$\frac{\sin\beta'.\sin\gamma'}{\sin\beta.\sin\gamma} + \frac{\sin\gamma'.\sin\alpha'}{\sin\gamma.\sin\alpha} + \frac{\sin\alpha'.\sin\beta'}{\sin\alpha.\sin\beta} = 1,$$

regard being had to the signs as well as the magnitudes of the six angles involved.

For, as in Cor. 5°. of the preceding article, drawing through any arbitrary point O, four lines OA, OB, OC, OD parallel to a, b, c, d; then since $BOC = \alpha$, $COA = \beta$, $AOB = \gamma$, and $AOD = \alpha'$, $BOD = \beta'$, $COD = \gamma'$, the relation is evident from that of Cor. 5°.

Cor. 8°. *If A, B, C be the three vertices of any triangle, and AX, BY, CZ three parallels drawn from them in any direction to meet the three opposite sides BC, CA, AB, then always*

$$\frac{BX.CX}{AX^2} + \frac{CY.AY}{BY^2} + \frac{AZ.BZ}{CZ^2} = 1,$$

regard being had to the signs as well as the magnitudes of the three rectangles involved.

For, if α, β, γ be the three angles of the triangle at A, B, C, and α', β', γ' those at which the three opposite sides intersect

any line parallel to the common direction of the three parallels, since then

$$\frac{BX}{AX} = \frac{\sin\gamma'}{\sin\beta} \text{ and } \frac{CX}{AX} = \frac{\sin\beta'}{\sin\gamma};$$

therefore
$$\frac{BX.CX}{AX^2} = \frac{\sin\beta'.\sin\gamma'}{\sin\beta.\sin\gamma},$$

and similarly,

$$\frac{CY.AY}{BY^2} = \frac{\sin\gamma'.\sin\alpha'}{\sin\gamma.\sin\alpha} \text{ and } \frac{AZ.BZ}{CZ^2} = \frac{\sin\alpha'.\sin\beta'}{\sin\alpha.\sin\beta},$$

and the relation consequently is evident from that of Cor. 7°.

84. We shall conclude the present chapter with one or two applications of a very simple problem of very frequent occurrence in Pure and Applied Geometry.

Given in magnitude and sign the ratio $m : n$ of the segments AP and BP into which a given line AB is cut at a point P, to determine the segments in magnitude and sign.

Since, by hypothesis, $AP : BP = m : n$, therefore

$$AP : AP - BP : m : m - n, \text{ and } BP : BP - AP = n : n - m,$$

and since in all cases $AP - BP = AB$, and $BP - AP = BA$, therefore

$$AP = \frac{m}{m-n}.AB = \frac{m}{n-m}.BA,$$

$$BP = \frac{n}{m-n}.AB = \frac{n}{n-m}.BA,$$

which are the general formulæ by which to calculate in numbers the segments of a line of given length cut in any given ratio.

Cor. 1°. As an application of the preceding let it be required *to determine for any triangle ABC the lengths of the bisectors, external and internal, of the three angles, and the segments they intercept on the opposite sides.*

If AX, BY, CZ be the three external, and AX', BY', CZ' the three internal bisectors, then since (Euc. VI. 3)

$$\frac{BX}{CX} = +\frac{BA}{CA} \text{ and } \frac{BX'}{CX'} = -\frac{BA}{CA};$$

therefore, by the above,

$$BX = \frac{BA}{BA - CA}.BC, \quad BX' = \frac{BA}{BA + CA}.BC,$$

$$CX = \frac{CA}{CA - BA}.CB, \quad CX' = \frac{CA}{CA + BA}.CB,$$

and therefore at once, by subtraction, remembering that similar formulæ for the same reason hold for the other two sides,

$$X'X = \frac{2BA.CA}{BA^2 - CA^2}.BC, \quad Y'Y = \frac{2CB.AB}{CB^2 - AB^2}.CA,$$

$$Z'Z = \frac{2AC.BC}{AC^2 - BC^2}.AB,$$

which are the general formulæ by which to calculate in numbers the lengths of the three intercepts $X'X$, $Y'Y$, $Z'Z$, when the sides of the triangle are given.

Since again, at once, by multiplication,

$$BX.CX = \frac{BA.CA}{(BA - CA)^2}.BC^2, \text{ and } BX'.CX' = -\frac{BA.CA}{(BA + CA)^2}.BC^2,$$

with corresponding values for the other two sides, therefore, by Cor. 2°. (83),

$$AX^2 = BA.CA\left\{\left(\frac{BC}{BA - CA}\right)^2 - 1\right\},$$

and

$$AX'^2 = BA.CA\left\{1 - \left(\frac{BC}{BA + CA}\right)^2\right\},$$

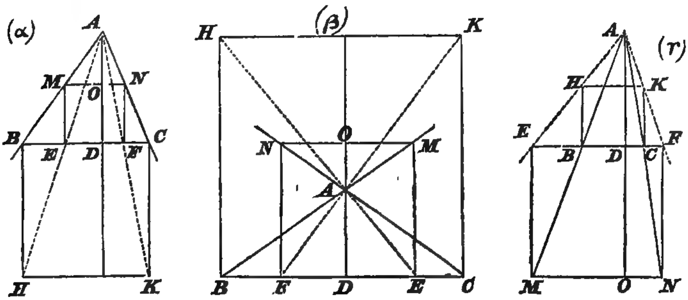
which with similar values for the other two sides are the formulæ by which to calculate in numbers the lengths of the six bisectors AX and AX', BY and BY', CZ and CZ' when the sides of the triangle are given.

From the above values for $X'X$, $Y'Y$, $Z'Z$, it is evident that their reciprocals are connected in all cases by the two following relations:

$$\frac{1}{XX'} + \frac{1}{YY'} + \frac{1}{ZZ'} = 0, \text{ and } \frac{BC^2}{XX'} + \frac{CA^2}{YY'} + \frac{AB^2}{ZZ'} = 0,$$

from which, regarding them as positive or negative according as they are similar or opposite in direction with the sides of the triangle measured from B to C, from C to A, and from A to B respectively, it is evident that one of them must have in all cases the sign opposite to that of the other two.

COR. 2°. As a second application of the same, let it be required *to determine for any triangle ABC the sides of the squares exscribed and inscribed to the three sides, and the segments they intercept on the perpendiculars from the opposite vertices.*



Let $EFMN$ be, fig. α, the inscribed, or, figs. β and γ, the exscribed square corresponding to the side BC of the triangle BAC; then drawing AD the perpendicular on that side from the opposite vertex A, intersecting MN in O, by similar triangles MAN and BAC, we have $MN : AO = BC : AD$, but, on account of the square, $MN = OD$, therefore, disregarding signs for a moment, $DO : AO = BC : AD$; that is the perpendicular AD is cut at the point O, internally, fig. α, in the case of the inscribed, and externally, figs. β and γ, in the case of the exscribed square in the ratio of $BC : AD$; and therefore, by the above

$$OD = \frac{BC}{BC \pm AD} . AD = \frac{BC.AD}{BC \pm AD},$$

the upper sign corresponding to the inscribed and the lower to the exscribed square.

Similar formulæ holding of course for the other two sides CA and AB; if a, b, c be the three sides of the triangle, p, q, r the three perpendiculars upon them from the opposite vertices, x, y, z the sides of the three inscribed, and x', y', z' those of the three exscribed squares; then, by the above,

$$x = \frac{ap}{a+p}, \quad y = \frac{bq}{b+q}, \quad z = \frac{cr}{c+r},$$

$$x' = \frac{ap}{a-p}, \quad y' = \frac{bq}{b-q}, \quad z' = \frac{cr}{c-r},$$

which are the general formulæ by which to calculate in numbers the sides of the six squares when the sides of the triangle are given.

It is evident from these formulæ, or directly, that while the inscribed square corresponding to any side of a triangle lies always on the same side of that side with the triangle itself, (fig. α); the exscribed square on the contrary lies on the same or on the opposite side, figs. β and γ, according as the side of the triangle to which it corresponds is greater or less than the perpendicular upon it from the opposite vertex; in the particular case when a side of a triangle is equal to the perpendicular upon it from the opposite vertex, the exscribed square corresponding to such side is infinite, and may therefore be regarded as lying indifferently in either direction.

Combining the above formulæ in corresponding pairs, by addition and subtraction, we have immediately

$$x'+x=\frac{2ap}{a^2-p^2}.a,\quad y'+y=\frac{2bq}{b^2-q^2}.b,\quad z'+z=\frac{2cr}{c^2-r^2}.c,$$

$$x'-x=\frac{2ap}{a^2-p^2}.p,\quad y'-y=\frac{2bq}{b^2-q^2}.q,\quad z'-z=\frac{2cr}{c^2-r^2}.r,$$

which latter, regard being had to their signs as well as their magnitudes, are the formulæ for the lengths of the segments intercepted on the three perpendiculars of the triangle by the three pairs of squares.

Taking again the reciprocals of the above formulæ, viz.

$$\frac{1}{x}=\frac{1}{p}+\frac{1}{a},\quad \frac{1}{y}=\frac{1}{q}+\frac{1}{b},\quad \frac{1}{z}=\frac{1}{r}+\frac{1}{c},$$

$$\frac{1}{x'}=\frac{1}{p}-\frac{1}{a},\quad \frac{1}{y'}=\frac{1}{q}-\frac{1}{b},\quad \frac{1}{z'}=\frac{1}{r}-\frac{1}{c},$$

and combining them also in corresponding pairs, by addition and subtraction, we get

$$\frac{1}{x}+\frac{1}{x'}=\frac{2}{p},\quad \frac{1}{y}+\frac{1}{y'}=\frac{2}{q},\quad \frac{1}{z}+\frac{1}{z'}=\frac{2}{r},$$

$$\frac{1}{x}-\frac{1}{x'}=\frac{2}{a},\quad \frac{1}{y}-\frac{1}{y'}=\frac{2}{b},\quad \frac{1}{z}-\frac{1}{z'}=\frac{2}{c},$$

which are the formulæ by which to calculate in numbers a side and perpendicular of a triangle when their inscribed or exscribed squares are given.

From the several preceding formulæ it is evident that any two of the four corresponding magnitudes, viz., a side of a triangle, the perpendicular upon it, the inscribed and exscribed squares resting upon it, determine the other two.

The sides of the squares inscribed and exscribed to any side BC of a triangle ABC, being given by the above formulæ, the squares themselves can of course be immediately constructed; if however it were required only to construct them without having also to calculate their sides, of the several methods of doing so the following is perhaps the most convenient.

On the side BC of the triangle upon which the squares are to be constructed, describe the square $BCHK$, and connect its two opposite vertices H and K with the opposite vertex A of the triangle; the two connecting lines HA and KA will intercept on BC the base EF of the required inscribed or exscribed square $EFMN$—of the inscribed if HK and A lie at opposite sides of BC (fig. α)—of the exscribed if they lie at the same side of it (figs. β and γ).

For, drawing EM and FN perpendiculars to BC and joining MN; as the three lines AH, AK, and AB pass through a point A, and as EM and EF are parallels to HB and HK, therefore $EM : EF = HB : HK$, and similarly $FN : FE = KC : KH$, but by construction $HB = KC = HK$, therefore $EM = FN = EF$, and therefore &c.

A method exactly similar might obviously be employed to solve the more general problem: "*On any side BC of a given triangle ABC to inscribe or exscribe a parallelogram of any given form.*"

CHAPTER VI.

THEORY GENERAL OF THE MEAN CENTRE OF ANY SYSTEM OF POINTS FOR ANY SYSTEM OF MULTIPLES.

85. The main features of this theory for the particular case of a system of points disposed along a line having been already given in sections 79, 80, 81 of the preceding, its extension to a system of points disposed in any manner will form the chief subject of the present chapter; the following fundamental theorem may be regarded as the basis of this extension.

If A, B, C, D, &c. be any system of points, disposed in any manner, but none infinitely distant, a, b, c, d, &c. any system of corresponding multiples, positive or negative, but none infinitely great, and O a point such that for two lines M and N passing through it $\Sigma(a.AM)=0$, and $\Sigma(a.AN)=0$; then for every line L passing through O $\Sigma(a.AL)=0$, regard being had in all the sums to the signs as well as the magnitudes of the several quantities involved.

For, if O be at an infinite distance, then for the several points by Cor. 5°. (Art. 82) of the preceding chapter,

$$MN.AL+NL.AM+LM.AN=0,$$
$$MN.BL+NL.BM+LM.BN=0,$$
$$MN.CL+NL.CM+LM.CN=0,$$
$$MN.DL+NL.DM+LM.DN=0, \text{ \&c.}$$

And, if O be at a finite distance, then for the several points by Cor. 6°. (Art. 82) of the same,

$$\sin MN.AL+\sin NL.AM+\sin LM.AN=0,$$
$$\sin MN.BL+\sin NL.BM+\sin LM.BN=0,$$
$$\sin MN.CL+\sin NL.CM+\sin LM.CN=0,$$
$$\sin MN.DL+\sin NL.DM+\sin LM.DN=0, \text{ \&c.}$$

which multiplied in either case by a, b, c, d, &c. and added, give at once, in the former case the relation

$$MN.\Sigma(a.AL) + NL.\Sigma(a.AM) + LM.\Sigma(a.AN) = 0,$$

and in the latter case the relation

$$\sin MN.\Sigma(a.AL) + \sin NL.\Sigma(a.AM) + \sin LM.\Sigma(a.AN) = 0,$$

from which it follows immediately in either case that if any two of the three sums $\Sigma(a.AL)$, $\Sigma(a.AM)$, $\Sigma(a.AN) = 0$, the third also $= 0$, and therefore &c.

The case of O at an infinite distance corresponds, as may be easily shewn, to that of $\Sigma(a) = 0$, a case requiring, as we shall see, special treatment in almost every point connected with the present subject; for, since $\Sigma(a.AN) - \Sigma(a.AM) = \Sigma(a).MN$ for every two parallel lines M and N whatever be their interval of separation MN; therefore if, as above, $\Sigma(a.AN) = \Sigma(a.AM)$ for any two parallel lines M and N not coinciding with each other, then $\Sigma(a) = 0$, and if conversely $\Sigma(a) = 0$, then $\Sigma(a.AN) = \Sigma(a.AM)$ for every two parallel lines M and N not infinitely distant from each other, and therefore &c.

86. The point O related as above to a system of points A, B, C, D, &c. that for every line L passing through it the sum

$$a.AL + b.BL + c.CL + d.DL + \&c. = 0,$$

is termed *the centre of mean position*, or more shortly *the mean centre* of the system of points for the system of multiples a, b, c, d, &c. and is in general a unique point depending upon and varying with the positions of the points and the values of the multiples; the propriety of the name depending on the properties of the point will appear in the sequel.

In the science of Mechanics, if A, B, C, D, &c. be the positions, and a, b, c, d, &c. the masses of any system of material particles situated in the same plane, then is the point O, as above defined, *the centre of gravity* of the system; in that science, therefore, all propositions relating to this subject are of considerable importance.

87. For every system of points A, B, C, D, &c. there exists a particular system of multiples a, b, c, d, &c. indeterminate of

course in absolute but fixed and unique in relative values, such that for *every* line L not actually at infinity, the sum $\Sigma(a.AL)=0$, and for which therefore *the mean centre O of the system is indeterminate;* in all such cases it is easy to see, 1°. *that* $\Sigma(a)=0$, and 2°. *that each point of the system is the mean centre of the others for their respective multiples;* for, the values of $\Sigma(a.AL)$ being by hypothesis $=0$ for two different lines passing through a point at infinity, therefore by the preceding $\Sigma(a)=0$, and being again by hypothesis $=0$ for two different lines passing through any point of the system, therefore by the same that point is the mean centre of the others for their respective multiples; instances of such cases are of course exceptional, but whenever they present themselves, as they occasionally do, their exceptional peculiarities must always be attended to.

88. From the fundamental property of the preceding article, it is easy to see that if a system of multiples a, b, c, d, &c. corresponding to a system of points A, B, C, D, &c. be such that for any *three* lines L, M, N *not passing through a common point* $\Sigma(a.AL)=0$, $\Sigma(a.AM)=0$, $\Sigma(a.AN)=0$, then for *every* line I not actually at infinity $\Sigma(a.AI)=0$. For, if L', M', N' be any three lines passing respectively through the three points MN, NL, LM, and intersecting on I, then since by (85),

$$\Sigma(a.AL')=0,\quad \Sigma(a.AM')=0,\quad \Sigma(a.AN')=0,$$

therefore by the same $\Sigma(a.AI)=0$, and therefore &c.

89. From the same again it appears, that if for a system of multiples a, b, c, d, &c. a system of points A, B, C, D, &c. have *two* different mean centres O_1 and O_2, then is *every* point O indifferently a mean centre of the same system of points for the same system of multiples; for, whatever be the position of O, since for the two lines L_1 and L_2 connecting it with O_1 and O_2, the two sums $\Sigma(a.AL_1)$ and $\Sigma(a.AL_2)$ are both $=0$, therefore for every line L passing through O the sum $\Sigma(a.AL)=0$, and therefore &c. *Hence, whatever be the positions of the points A, B, C, D, &c. and whatever be the values of the multiples a, b, c, d, &c. the mean centre O is always either indeterminate or unique.*

90. *If A, B, C, D, &c. be the several vertices of a regular polygon of any order, and O the geometric centre of the figure, then is O the mean centre of the several points A, B, C, D, &c. for the particular system of multiples each = unity.*

For, if the polygon be of an even order, since for every line passing through O the several pairs of perpendiculars from pairs of opposite vertices are equal and opposite, therefore for every line passing through O the sum of the perpendiculars from all the vertices = 0, and therefore &c.; and, if the polygon be of an odd order, since for every line passing through O and through a vertex of the figure the several pairs of perpendiculars from pairs of vertices equidistant from that through which the line passes are equal and opposite, and the one from that vertex itself = 0, therefore for every line passing through O and through a vertex of the figure, and therefore by the preceding for every line passing through O, the sum of the perpendiculars from all the vertices = 0, and therefore &c.

In consequence of the above, all properties true in general of the mean centre of any system of points A, B, C, D, &c. for any system of multiples a, b, c, d, &c. whose sum is not = 0, are true in particular of the geometric centre of any regular polygon regarded as the mean centre of its several vertices for the particular system of multiples each = unity.

91. *If A, B, C be the three vertices of any triangle, and O their mean centre for any three multiples a, b, c, then always—*

1°. *The three lines AO, BO, CO intersect with the three opposite sides BC, CA, AB at three points X, Y, Z such that*

$$b.BX + c.CX = 0, \quad c.CY + a.AY = 0, \quad a.AZ + b.BZ = 0.$$

2°. *The three triangles BOC, COA, AOB are connected with the three multiples a, b, c by the proportions*

$$\text{area}BOC : \text{area}COA : \text{area}AOB = a : b : c,$$

regard being had to the signs as well as the magnitudes of the several quantities involved in each.

To prove 1°. Since for every three lines L, M, N passing through O, (86)

$$a.AL + b.BL + c.CL = 0, \quad a.AM + b.BM + c.CM = 0,$$
$$a.AN + b.BN + c.CN = 0,$$

if L pass through A, then

$$AL = 0 \text{ and } BL : CL = BX : CX,$$

and therefore $b.BX + c.CX = 0$; if M pass through B, then

$$BM = 0 \text{ and } CM : AM = CY : AY,$$

and therefore $c.CY + a.AY = 0$; and if N pass through C, then

$$CN = 0 \text{ and } AN : BN = AZ : BZ,$$

and therefore $a.AZ + b.BZ = 0$.

To prove 2°. Since the two triangles AOB and AOC have a common base AO, therefore

$$\text{area} AOB : \text{area} AOC = BL : CL = BX : CX,$$

since the two BOC and BOA have a common base BO, therefore

$$\text{area} BOC : \text{area } BOA = BM : CM = BY : CY,$$

and since the two COA and COB have a common base CO, therefore

$$\text{area } COA : \text{area} COB = AN : BN = AZ : BZ;$$

and the proportions 2°. follow therefore immediately from the relation 1°.

Cor. The above relations supply each an obvious method of determining the mean centre O of any three points A, B, C forming a triangle, for any three multiples a, b, c given in magnitude and sign; the two following particular cases are deserving of attention:

1°. If in absolute magnitude $a = b = c$, then AX, BY, CZ bisect the three sides of the triangle, all internally or two externally and one internally according as the signs of a, b, c are all similar or two opposite to the third; O in either case is the intersection of the three bisectors; and the three areas BOC, COA, AOB are equal in absolute magnitude and have signs in accordance with those of a, b, c.

2°. If in absolute magnitude $a : b : c = BC : CA : AB$, then AX, BY, CZ bisect the three angles of the triangle, all internally or two externally and one internally according as the signs of a, b, c are all similar or two opposite to the third; O in either case is the intersection of the three bisectors, and therefore the centre of the inscribed or of one of the three exscribed circles of the triangle; and the three areas BOC, COA, AOB are proportional in absolute magnitude to the three sides BC, CA, AB, and have signs in accordance with those of a, b, c.

92. *If A, B, C, D, &c. be any system of points, O their mean centre for any system of multiples a, b, c, d, &c. whose sum is not* $= 0$, *and L any arbitrary line, then always whatever be the position of L*

$$\Sigma (a . AL) = \Sigma (a) . OL,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, drawing through O the line M parallel to L, then since for any two parallel lines L and M whatever be their common direction or distance asunder $\Sigma (a . AL) - \Sigma (a . AM) = \Sigma (a) . ML$, if, as in the present case, one of them M passes through O, since for it $\Sigma (a . AM) = 0$ (86) therefore for the other L whatever be its position $\Sigma (a . AL) = \Sigma (a) . OL$, and therefore &c.

Cor. 1°. This is the property which gives to the point O its designation of "Mean Centre" of the system of points A, B, C, D, &c. for the system of multiples a, b, c, d, &c., and by its aid when the latter are both given the former may be determined in all cases by the following general construction:

Drawing arbitrarily any two lines L and L' not parallel to each other, the two parallels to them M and M' distant from them by the intervals LM and $L'M'$ equal in magnitude and sign to the quantities $\dfrac{\Sigma (a . LA)}{\Sigma (a)}$ and $\dfrac{\Sigma (a . L'A)}{\Sigma (a)}$ pass, by the above, through, and therefore intersect at, the mean centre O; in the particular case where $\Sigma (a) = 0$, the position of O thus given is at infinity (85), *unless also* $\Sigma (a . LA)$ *and* $\Sigma (a . L'A)$ *both* $= 0$, *in which exceptional case it is indeterminate* (87).

Cor. 2°. The mean centre O of any given system of points A, B, C, D, &c. for any given system of multiples a, b, c, d, &c. may also be determined by the following in general less rapid, but in many cases not less convenient process, based like that just given on the above, viz.:

Connect any two points A and B of the system, and take on the connecting line AB the point P for which $a . AP + b . BP = 0$ (77). Connect then the point P with any third point C of the system, and take on the connecting line PC the point Q for which $(a + b) . PQ + c . CQ = 0$. Connect then the point Q with any fourth point D of the system, and take on the connecting line QD the point R for which $(a + b + c) . QR + d . DR = 0$. Connect then the point R with any fifth point E of the system,

and take on the connecting line RE the point S for which $(a+b+c+d).RS+e.ES=0$, and so on, until all the points of the system are exhausted, the last point O thus determined is the mean centre required.

For since for every arbitrary line L, by (77) Cor.

$$a.AL+b.BL=(a+b).PL,$$
$$(a+b).PL+c.CL=(a+b+c).QL,$$
$$(a+b+c).QL+d.DL=(a+b+c+d).RL,$$
$$(a+b+c+d).RL+e.EL=(a+b+c+d+e).SL, \text{ \&c.}$$

therefore for the last point O, by addition

$$a.AL+b.BL+c.CL+d.DL+\text{\&c.}=(a+b+c+d+\text{\&c.}).OL,$$

which, by the above, is the characteristic property of the mean centre.

In the particular case when $\Sigma(a)=0$, the point O thus determined being the point of external bisection of the last connecting line in the above process is therefore at infinity, *unless when the length of that connecting line* $=0$ *in which exceptional case it is indeterminate.*

Cor. 3°. Stating the above general relation in the equivalent form $\Sigma(a.AL)-\Sigma(a).OL=0$, it appears that, if to any system of points A, B, C, D, &c. be added their mean centre O for any system of multiples a, b, c, d, &c., then is the system of points A, B, C, D, &c. and O, for the system of multiples a, b, c, d, &c. and $-\Sigma(a)$, of the exceptional character mentioned in (87), for which for *every* line L not at infinity the sum $\Sigma(a.AL)=0$, and for which therefore the mean centre is *indeterminate.* Hence the original system of points A, B, C, D, &c. and of multiples a, b, c, d, &c. being entirely arbitrary, it appears that—

For a system of the exceptional character whose mean centre is indeterminate, all but one of the points may have any positions whatever, and their corresponding multiples any values whatever, provided only the remaining point be the mean centre of the others for their system of multiples, and the remaining multiple corresponding to it be equal in magnitude and opposite in sign to the sum of the others.

Cor. 4°. Since for every line L tangent to any circle round O as centre the distance OL is constant and equal the radius of the circle, and since, by the above, the sum $\Sigma(a.AL)$ is con-

stant when the radius OL is constant, and conversely, therefore—

If A, B, C, D, &c. be any system of points, and O their mean centre for any system of multiples a, b, c, d, &c. whose sum is not $= 0$, then for every line L tangent to any circle round O as centre the sum $\Sigma(a.AL)$ is constant and = the radius of the circle multiplied by $\Sigma(a)$, and, conversely, every line L for which the sum $\Sigma(a.AL)$ is constant touches the circle round O as centre whose radius = the constant sum divided by $\Sigma(a)$.

This property supplies obvious solutions of the following general problems, viz.: "Given any system of points A, B, C, D, &c., and any system of corresponding multiples a, b, c, d, &c. whose sum is not $= 0$, to draw a line L parallel to a given line, or passing through a given point, or touching a given circle, so that the sum $\Sigma(a.AL)$ shall $= 0$, or be a maximum, or have any given value."

Cor. 5°. *For every line L tangent to the circle inscribed in any triangle ABC the sum of the three rectangles*

$$BC.AL + CA.BL + AB.CL$$

is constant and equal to double the area of the triangle.

For, by (91), the centre O of that circle being the mean centre of the three points A, B, C for the three multiples BC, CA, AB, therefore, by the above,

$$BC.AL + CA.BL + AB.CL = (BC + CA + AB).OL;$$

but

$$BC.OL = 2\,\text{area}\,BOC,\ CA.OL = 2\,\text{area}\,COA,\ AB.OL = 2\,\text{area}\,AOB;$$

therefore their sum $= 2$ area ABC, and therefore &c.

A relation exactly similar holds of course for each of the three exscribed circles of the triangle, the sign of the side to which the circle is exscribed being merely changed in the above, see 91, Cor.

Cor. 6°. *For every line L tangent to any circle concentric with a regular polygon of any order n the sum of the perpendiculars from the several vertices is constant and $= n$ times the radius of the circle.*

For, by (90), the centre O of the polygon being the mean centre of the several vertices A, B, C, D, &c. for the particular system of multiples each $= 1$, therefore, by the above, $\Sigma(AL) = n.OL$, and therefore &c.

For a regular polygon of any order n the sum of the perpendiculars from any point P upon the several sides is also constant and = n times the radius of the circle inscribed in the figure.

For the sums of the perpendiculars from the centre O and from any other point P upon the several sides multiplied each by the common length of all the sides = double the area of the figure, and therefore &c.

93. *If any system of points $\Sigma(A)$ and of corresponding multiples $\Sigma(a)$ be divided into any number of groups $\Sigma(A_1)$, $\Sigma(A_2)$, $\Sigma(A_3)$, $\Sigma(A_4)$, &c., and $\Sigma(a_1)$, $\Sigma(a_2)$, $\Sigma(a_3)$, $\Sigma(a_4)$, &c., none of the latter being $=0$; then, if O_1, O_2, O_3, O_4, &c. be the several mean centres of the several groups of points for the several groups of corresponding multiples, the mean centre O of the system of points O_1, O_2, O_3, O_4, &c. for the systems of multiples $\Sigma(a_1)$, $\Sigma(a_2)$, $\Sigma(a_3)$, $\Sigma(a_4)$, &c. is the same as that of the system of points A, B, C, D, &c. for the system of multiples a, b, c, d, &c.*

For, since for every arbitrary line L, by the preceding,

$$\Sigma(a_1.A_1L)=\Sigma(a_1).O_1L,\quad \Sigma(a_2.A_2L)=\Sigma(a_2).O_2L,$$
$$\Sigma(a_3.A_3L)=\Sigma(a_3).O_3L,\quad \Sigma(a_4.A_4L)=\Sigma(a_4).O_4L,\ \&c.;$$

therefore the sum of all the first members = the sum of all the second members; but, by hypothesis, the sum of all the first members $=\Sigma(a.AL)$, and, by the preceding, the sum of all the second members

$$=\{\Sigma(a_1)+\Sigma(a_2)+\Sigma(a_3)+\Sigma(a_4)+\&c.\}.OL;$$

from which, since by hypothesis

$$\Sigma(a_1)+\Sigma(a_2)+\Sigma(a_3)+\Sigma(a_4)+\&c.=\Sigma(a);$$

therefore $\Sigma(a.AL)=\Sigma(a).OL$, and therefore &c.

COR. In the particular case when $\Sigma(a)=0$, if $\Sigma(A)$ be divided into any two groups $\Sigma(A_1)$ and $\Sigma(A_2)$ for which $\Sigma(a_1)$ and $\Sigma(a_2)$ are not separately $=0$; then since, by hypothesis, $\Sigma(a_1)+\Sigma(a_2)=0$, if O_1 and O_2 be the mean centres of the two groups for their respective shares of the multiples, that of the entire system for all the multiples being, by the above, the point of external bisection of the line O_1O_2 is therefore the unique point in which that line intersects infinity (15), *except only when the two partial mean centres O_1 and O_2 coincide in*

which exceptional case it is indeterminate (87). The division of $\Sigma(A)$ may, if we please, be into the two groups $\Sigma(A_+)$ and $\Sigma(A_-)$ corresponding to the division of $\Sigma(a)$ into its positive and negative constituents $\Sigma(a_+)$ and $\Sigma(a_-)$ respectively; or one group may, if we please, consist of but a single point and the other of all the rest.

94. *If A, B, C, D, &c. be any system of points, M any line parallel to the direction of their infinitely distant mean centre for any system of multiples a, b, c, d, &c. whose sum $=0$, and L any other line, then, whatever be the position of L,*

$$\Sigma(a.AL) = k.\sin ML,$$

k being a constant depending only on the disposition of the points and the values of the multiples.

For, if N be a third line passing through the intersection P of L and M, and perpendicular to the latter, then as in (85) the three lines LMN passing through a common point P,

$$\sin MN.\Sigma(a.AL) + \sin NL.\Sigma(a.AM) + \sin LM.\Sigma(a.AN) = 0,$$

from which as $\Sigma(a.AM)=0$ from the property of the mean centre (86), and as $\sin MN = 1$ from the right angle MN (60), therefore

$$\Sigma(a.AL) = \Sigma(a.AN).\sin ML,$$

which proves the proposition, the two sums $\Sigma(a.AL)$ and $\Sigma(a.AN)$ depending when $\Sigma(a)=0$ (85) only on the directions and not on the absolute positions of L and N.

Otherwise thus, as a corollary from the general case, when $\Sigma(a)$ is not $=0$; conceiving the entire system of points $\Sigma(A)$ divided into any two groups $\Sigma(A_1)$ and $\Sigma(A_2)$ for which the sums $\Sigma(a_1)$ and $\Sigma(a_2)$ of the corresponding groups of multiples are not separately $=0$; then, by the general relation of the preceding article, if O_1 and O_2 be the mean centres of the two partial groups for their respective systems of multiples, and L any line intersecting O_1O_2 at any point P and at any angle α,

$$\Sigma(a_1.A_1L) = \Sigma(a_1).O_1L, \text{ and } \Sigma(a_2.A_2L) = \Sigma(a_2).O_2L,$$

and therefore, by addition,

$$\Sigma(a.AL) = \Sigma(a_1).O_1L + \Sigma(a_2).O_2L;$$

but $O_1L = O_1P.\sin\alpha$, $O_2L = O_2P.\sin\alpha$, and $\Sigma(a_1) + \Sigma(a_2) = 0$;

therefore

$$\Sigma(a.AL) = \Sigma(a_1).O_1O_2.\sin\alpha, \text{ or } \Sigma(a_2).O_2O_1.\sin\alpha,$$

which proves the proposition, and gives at the same time in its most convenient form the value of the constant k or $\Sigma(a.AN)$ viz. $\Sigma(a_1).O_1O_2$ or $\Sigma(a_2).O_2O_1$. See (81).

The law of the variation of the sum $\Sigma(a.AL)$ for different positions of L is therefore very simple when $\Sigma(a) = 0$; depending only on the direction and not on the absolute position of L; vanishing for the direction of the infinitely distant mean centre of the system; being a maximum for the rectangular direction; and varying as the sine of the angle of inclination to the central for every intermediate direction; *in the exceptional case where the two partial mean centres O_1 and O_2 coincide, and when* (93, Cor.) *the position of O is consequently indeterminate, the sum $\Sigma(a.AL)$ undergoes no variation and is absolutely $= 0$ for every position of L not actually at infinity*, see (87).

COR. 1°. By means of the above relation the direction of the infinitely distant mean centre of a given system of points A, B, C, D, &c. for a given system of multiples a, b, c, d, &c. whose sum $= 0$, if not previously known may be readily determined. For drawing arbitrarily any two lines L and L' not parallel to each other, the line M dividing the angle between them LL' so that in magnitude and sign

$$\sin ML : \sin ML' = \Sigma(a.AL) : \Sigma(a.AL')$$

gives, by the above, the required direction; *in the exceptional case when the two sums $\Sigma(a.AL)$ and $\Sigma(a.AL')$ are both $= 0$, the mean centre of the system is indeterminate*, see (87).

COR. 2°. The above relation also supplies obvious solutions of the six following problems, viz.: "given any system of points A, B, C, D, &c., and any system of corresponding multiples a, b, c, d, &c. whose sum $= 0$, to draw a line L passing through a given point or touching a given circle so that the sum $\Sigma(a.AL)$ shall be nothing, or a maximum, or have any given value."

95. *If any system of points A, B, C, D, &c. and their mean centre O for any system of multiples a, b, c, d, &c. be projected in any common direction upon any line L, the projection O' of the*

mean centre is always the mean centre of the projections A', B', C', D', &c. of the several points for the same system of multiples.

For, from the several points A, B, C, D, &c. conceiving lines AA_1, BB_1, CC_1, DD_1, &c. drawn parallel to the line L to meet the line OO'; then since, Euc. I. 34, $AA_1 = A'O'$, $BB_1 = B'O'$, $CC_1 = C'O'$, $DD_1 = D'O'$, &c., and since, by the fundamental property of the mean centre (86), $\Sigma(a.AA_1) = 0$, therefore $\Sigma(a.A'O') = 0$, and therefore O' is the mean centre of the system of points A', B', C', D', &c. for the system of multiples a, b, c, d, &c.; when L passes through O then $\Sigma(a.A'O) = 0$ and O itself is the mean centre of the projected as well as of the original system for the same system of multiples.

In the particular case when O is at an infinity, and when therefore $\Sigma(a) = 0$, its projection O' upon every base L is of course also at infinity, *except only when the direction of projection is parallel to that of O itself in which case it is indeterminate.*

In the exceptional case when O itself is indeterminate, and when therefore again $\Sigma(a) = 0$, its projection O' upon every base and for every direction of projection is of course also indeterminate.

96. *If A, B, C, D, &c. be any system of points, O their mean centre for any system of multiples a, b, c, d, &c. whose sum is not $= 0$, and L and M any two parallel lines, then always*

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = \Sigma(a).(OL^2 - OM^2),$$

whatever be the common direction and distance asunder of L and M.

For, identically,

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = \Sigma(a).\{(AL + AM).(AL - AM)\},$$

from which since $(AL - AM) =$ the constant interval between L and $M = (OL - OM)$, and since, by (92), $\Sigma(a.AL) = \Sigma(a).OL$, and $\Sigma(a.AM) = \Sigma(a).OM$, therefore at once

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = \Sigma(a).(OL + OM).(OL - OM)$$
$$= \Sigma(a).(OL^2 - OM^2). \quad \text{Q.E.D.}$$

Cor. 1°. When one of the lines M passes through O, then for the other L,

$$\Sigma(a.AL^2) = \Sigma(a.AM^2) + \Sigma(a).OL^2,$$

from which it appears that for a given direction of L the sum

$\Sigma(a.AL^2)$ is a minimum when L passes through O, and has equal values for every two positions equidistant in opposite directions from O; it appears also from the same that if the sum $\Sigma(a.AM^2)$ is constant for all lines passing through O the sum $\Sigma(a.AL^2)$ is constant for all lines touching a circle of any radius described round O as centre.

COR. 2°. The same relation also supplies an obvious solution of the general problem: "Given any system of points A, B, C, D, &c., and any system of corresponding multiples a, b, c, d, &c., whose sum is not $=0$; to draw a line L in a given direction so that the sum $\Sigma(a.AL^2)$ shall be given."

97. *If A, B, C, D, &c. be any system of points, a, b, c, d, &c. any system of corresponding multiples whose sum $=0$, and L and M any two parallel lines, then always*

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = 2.k.\sin\alpha.ML$$

k and α having the same signification as in (94).

For, as in the preceding, identically

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = \Sigma(a).\{(AL+AM).(AL-AM)\}$$

from which since $(AL-AM) = ML$ and since (94)

$$\Sigma(a.AL) = \Sigma(a.AM) = k.\sin\alpha,$$

therefore at once, as above,

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = 2k.\sin\alpha.ML.$$

COR. 1°. When one of the lines M is the particular line for its direction for which the sum $\Sigma(a.AM^2) = 0$, then for the other L,

$$\Sigma(a.AL^2) = 2k.\sin\alpha.ML,$$

from which it appears that for a given direction of L the sum $\Sigma(a.AL)$ follows a very simple law of variation when $\Sigma(a)=0$; being simply proportional in sign as well as in magnitude to the distance of L from a certain line M in that direction; admitting therefore of no minimum or maximum value; passing through 0 and ∞ with the distance ML; and changing sign at the passage through each. *For the particular direction for which $\alpha=0$ whatever be the value of k, and for the exceptional case for which $k=0$ whatever be the value of α, the sum $\Sigma(a.AL^2)$ undergoes no variation with the movement of L, but*

preserves in magnitude and sign the same constant value for every position of L in the same constant direction.

COR 2°. To find the line M corresponding to a given direction of L, for which in the general case the sum $\Sigma(a.AM^2)=0$; drawing arbitrarily any line L in the given direction, the parallel M distant from it by the interval $ML =$ in magnitude and sign to the quantity $\frac{\Sigma(a.AL^2)}{2k.\sin\alpha} = \frac{\Sigma(a.AL^2)}{2\Sigma(a.AL)}$, by the relation of Cor. 1°, is that required. *For the particular direction for which* $\alpha = 0$ *whatever be the value of* k, *and for the exception case for which* $k=0$ *whatever be the value of* α, *the sum* $\Sigma(a.AL)$ *being* $=0$, *the position of* M *given by the above is at infinity, unless at the same time the sum* $\Sigma(a.AL^2)$ *also* $=0$ *in which case it is indeterminate.*

COR. 3°. The above supplies an obvious solution of the following general problem: "Given any system of points A, B, C, D, &c., and any system of corresponding multiples a, b, c, d, &c. whose sum $=0$, to draw a line L in any given direction so that the sum $\Sigma(a.AL^2)$ shall have a given magnitude and sign."

98. *If* A, B, C, D, *&c. be any system of points,* O *their mean centre for any system of multiples* a, b, c, d, *&c. whose sum is not* $=0$, *and* P *any arbitrary point, then always, whatever be the position of* P,

$$\Sigma(a.AP^2) = \Sigma(a.AO^2) + \Sigma(a).OP^2$$

the same relation as for a system of points on a line and leading to the same consequences. See (80).

For, from the several points A, B, C, D, &c. conceiving perpendiculars AA', BB', CC', DD' &c., let fall upon the line OP, then since (75, Cor. 2°.)

$$AP^2 = AO^2 + OP^2 + 2A'O.OP$$
$$BP^2 = BO^2 + OP^2 + 2B'O.OP$$
$$CP^2 = CO^2 + OP^2 + 2C'O.OP$$
$$DP^2 = DO^2 + OP^2 + 2D'O.OP, \text{ \&c.}$$

therefore multiplying by a, b, c, d, &c. and adding

$$\Sigma(a.AP^2) = \Sigma(a.AO^2) + \Sigma(a).OP^2 + 2.\Sigma(a.A'O).OP,$$

from which, since by (95), $\Sigma(a.A'O)=0$, therefore &c.

COR. 1°. *If round O as centre and with a radius OR whose square = the absolute value of* $\frac{\Sigma(a.AO^2)}{\Sigma(a)}$, *disregarding its sign, a circle be described intersecting the line OP at the two points M and N, and the perpendicular to it through O in either direction at the point S, then, whatever be the position of P, the sum* $\Sigma(a.AP^2) = \Sigma(a).SP^2$ *or* $= \Sigma(a).MP.NP$ *according as* $\Sigma(a)$ *and* $\Sigma(a.AO^2)$ *have similar or opposite signs.*

For, since, by the above relation,

$$\Sigma(a.AP^2) = \Sigma(a).OP^2 + \Sigma(a.AO^2),$$

and since, by construction, $\Sigma(a.AO^2) = \pm\Sigma(a).OR^2$, therefore

$$\Sigma(a.AP^2) = \Sigma(a).(OP^2 \pm OR^2) = \Sigma(a).SP^2 \text{ or } \Sigma(a).MP.NP.$$

Hence, in both cases, the variable sum $\Sigma(a.AP^2)$ has the same value for all positions of P equidistant from O, and the minimum value for the point O itself; it being remembered however that as it vanishes in the second case for all points on the circle OR, and increases negatively from the circumference in to the centre, the term minimum is to be understood in the sense of negative maximum in that case. See (80), Cor. 1°.

COR. 2°. *For every point P on any circle round O as centre, the sum* $\Sigma(a.AP^2)$ *is constant and* $= \Sigma(a.OA^2) + \Sigma(a).\text{radius}^2$ *of circle; and, conversely, every point P for which the sum* $\Sigma(a.AP^2)$ *is constant lies on the circle round O as centre the square of whose radius* $= \frac{\Sigma(a.AP^2) - \Sigma(a.AO^2)}{\Sigma(a)}$.

These are both evident from the above, the first from the general relation $\Sigma(a.AP^2) = \Sigma(a.AO^2) + \Sigma(a).OP^2$, and the other from its equivalent $\Sigma(a.AP^2) - \Sigma(a.AO^2) = \Sigma(a).OP^2$; and they supply obvious solutions of the six general problems, viz.: "Given any system of points A, B, C, D, &c. and any system of corresponding multiples a, b, c, d, &c. whose sum is not $= 0$, to determine on a given line or circle the point P for which the sum $\Sigma(a.AP^2)$ shall be a maximum, a minimum, or given."

From the general property of this corollary, combined with that of Cor. 4°. (92), it follows evidently that every circle round O as centre is at once the locus of a variable point P for which the sum $\Sigma(a.AP^2)$ is constant, and the envelope of a variable line L for which the sum $\Sigma(a.AL)$ is constant.

COR. 3°. *For every point P on the circle inscribed in any triangle ABC, the sum $BC.AP^2 + CA.BP^2 + AB.CP^2$ is constant, and exceeds the corresponding sum for the centre O by double the area of the triangle multiplied by the radius of the circle.*

For, by (91), the centre O of that circle being the mean centre of the three points A, B, C for the three multiples BC, CA, AB, therefore, by the above,

$$BC.AP^2 + CA.BP^2 + AB.CP^2 = BC.AO^2 + CA.BO^2 + AB.CO^2 + (BC + CA + AB).OP^2;$$

but, as in Cor. 5°., (92),

$$(BC + CA + AB).OP = 2 \text{ area of triangle};$$

and therefore &c.

A relation exactly similar holds of course for each of the three exscribed circles of the triangle, the sign of the side to which the circle is exscribed being merely changed in the above. See (91), Cor.

COR. 4°. *If O be the centre and OR the radius of the circle which passes through the several vertices A, B, C, D, &c. of a regular polygon of any order n, then for every point P without, within, or upon the circle $\Sigma(AP^2) = n.(OR^2 + OP^2)$.*

For O being the mean centre of the system of points A, B, C, D, &c. for the system of multiples each $= 1$ (90); therefore, by the above,

$$\Sigma(AP^2) = \Sigma(AO^2) + n.OP^2;$$

but $\qquad OA = OB = OC = OD, \text{ &c.} = OR,$

therefore $\qquad \Sigma(AP^2) = n.OR^2 + n.OP^2.$

In the particular case when P is on the circle, since then $OP = OR$, therefore $\Sigma(AP^2) = 2n.OR^2$.

99. *If A, B, C, D, &c. be any system of points, and O their mean centre for any system of multiples a, b, c, d, &c. whose sum is not $= 0$, then always*

$$\Sigma(a).\Sigma(a.AO^2) = \Sigma(ab.AB^2),$$

every binary combination of the points of the system being included in the latter summation.

For, in the general relation of the preceding article,

$$\Sigma(a.AP^2) = \Sigma(a).PO^2 + \Sigma(a.AO^2),$$

conceiving the arbitrary point P to coincide successively with the several points A, B, C, D, &c. of the system, then

$$a.AA^2+b.AB^2+c.AC^2+d.AD^2+\&c.=\Sigma(a).AO^2+\Sigma(a.AO^2),$$
$$a.BA^2+b.BB^2+c.BC^2+d.BD^2+\&c.=\Sigma(a).BO^2+\Sigma(a.AO^2),$$
$$a.CA^2+b.CB^2+c.CC^2+d.CD^2+\&c.=\Sigma(a).CO^2+\Sigma(a.AO^2),$$
$$a.DA^2+b.DB^2+c.DC^2+d.DD^2+\&c.=\Sigma(a).DO^2+\Sigma(a.AO^2),\&c.;$$

which multiplied by a, b, c, d, &c. and added give, as $AA=0$, $BB=0$, $CC=0$, $DD=0$, &c., the relation

$$\Sigma(ab.AB^2+ba.BA^2)=\Sigma(a).\Sigma(a.AO^2)+\Sigma(a).\Sigma(a.AO^2);$$

or, which is the same thing, the relation

$$2\Sigma(ab.AB^2)=2\Sigma(a).\Sigma(a.AO^2),$$

the same as the above multiplied by 2.

The relation just proved, as furnishing for any given system of points and multiples the value of the indispensable constant $\Sigma(a.AO^2)$ without requiring the previous determination of the point O, is, consequently, of considerable importance in every numerical application of the formulæ of the preceding article.

COR. 1°. *If O be the centre of the circle inscribed in any triangle ABC, then*

$$BC.AO^2+CA.BO^2+AB.CO^2=BC.CA.AB,$$

with similar relations for the centres of the three exscribed circles, the sign of the side corresponding to each being simply changed in the above.

For, by (91), O being the mean centre of the three vertices A, B, C for the three multiples BC, CA, AB, therefore, by the above,

$$(BC.AO^2+CA.BO^2+AB.CO^2).(BC+CA+AB)$$
$$=(BA.CA.BC^2+CB.AB.CA^2+AC.BC.AB^2)$$
$$=(BC.CA.AB).(BC+CA+AB);$$

which is the same as the above relation multiplied by

$$BC+CA+AB.$$

COR. 2°. *If O be the centre and OR the radius of the circle which passes through the several vertices A, B, C, D, &c. of a regular polygon of any order n, then always $\Sigma(AB^2)=n^2.OR^2$.*

For, by (90), O being the mean centre of the system of points A, B, C, D, &c. for the system of multiples each $=1$; therefore, by the above, $\Sigma(AB^2) = n.\Sigma(AO^2)$, but

$$OA = OB = OC = OD, \&c. = OR;$$

therefore $\Sigma(OA^2) = n.OR^2$, and therefore &c.

100. *If A, B, C, D, &c. be any system of points, M and N any two lines perpendicular to the direction of their infinitely distant mean centre O for any system of multiples a, b, c, d, &c. whose sum $=0$, and P and Q any two points on M and N not either of them at infinity, then always*

$$\Sigma(a.AP^2) - \Sigma(a.AQ^2) = 2k.NM,$$

k having the same signification as in (94).

For, drawing through P and Q two other lines M_0 and N_0 parallel to the direction of O, and therefore at right angles to M and N, then (Euc. I. 47)

$$\Sigma(a.AP^2) = \Sigma(a.AM^2) + \Sigma(a.AM_0^2),$$
$$\Sigma(a.AQ^2) = \Sigma(a.AN^2) + \Sigma(a.AN_0^2);$$

from which, by subtraction, remembering (97) that

$$\Sigma(a.AM^2) - \Sigma(a.AN^2) = 2k.NM,$$

and that $$\Sigma(a.AM_0^2) - \Sigma(a.AN_0^2) = 0,$$

the relation above stated is the immediate result.

Cor. 1°. From the relation just proved it follows that the two sums $\Sigma(a.AP^2)$ and $\Sigma(a.AQ^2)$ are both constant as long as the two points P and Q continue on the same two lines M and N perpendicular to the direction of O. If one of them N be the particular line in that perpendicular direction for every point Q of which the sum $\Sigma(a.AQ^2) = 0$, then for every point P on the other M not at infinity

$$\Sigma(a.AP^2) = 2k.NM = 2k.NP;$$

from which it appears that the sum $\Sigma(a.AP^2)$ follows, for different positions of P, a very simple law of variation when $\Sigma(a) = 0$; being simply proportional in sign as well as magnitude to the distance NP of the variable point P from a constant fixed line N perpendicular to the direction of O; admitting therefore of no minimum or maximum value; passing through nothing and infinity

with the distance NP; and changing sign at the passage through each. *In the exceptional case when* $k=0$, *and when therefore* (94) *the position of* O *is indeterminate, the sum* $\Sigma(a.AP^2)$ *undergoes no variation with the variation of* P, *but preserves in magnitude and sign the same constant value for all positions of* P *not actually at infinity;* an instance of which we have met with in (83), where for three points A, B, C on a line, we have seen that for the three multiples BC, CA, AB, the sum

$$BC.AP^2 + CA.BP^2 + AB.CP^2$$

is constant, whatever be the position of P provided only it be not at infinity.

COR. 2°. To find the particular line N perpendicular to the direction of O for every point of which in the general case the sum $\Sigma(a.AQ^2)=0$; drawing arbitrarily any line M perpendicular to the direction of O, the parallel to it N distant from it by the interval $NM=$ in magnitude and sign to the quantity

$$\frac{\Sigma(a.AM^2)}{2k} = \frac{\Sigma(a.AM^2)}{2\Sigma(a.AM)},$$

by the above is that required. *In the exceptional case when* $k=0$, *and when the direction of* L *is therefore indeterminate with that of* O, *the position of* N *given by the above is at infinity, unless at the same time* $\Sigma(a.AM^2)$ *also* $=0$ *in which case it is indeterminate.*

COR. 3°. The above supplies an obvious solution of the following general problem: "Given any system of points A, B, C, D, &c. and any system of corresponding multiples a, b, c, d, &c. whose sum $=0$, to determine on a given line or circle or any other figure the point or points P for which the sum $\Sigma(a.AP^2)$ shall have a given magnitude and sign."

101. The law, determined directly in the preceding, of the variation of $\Sigma(a.PA^2)$ for the particular case of $\Sigma(a)=0$, may also be inferred as a corollary from that of the same for the general case of $\Sigma(a)$ not $=0$, given in (98); for, as in (81) and (94), conceiving the entire system of points $\Sigma(A)$ divided into any two groups $\Sigma(A_1)$ and $\Sigma(A_2)$ for which the sums $\Sigma(a_1)$ and $\Sigma(a_2)$ of the corresponding groups of multiples are not

separately $=0$; then, by the general relation of that article (98), if O_1 and O_2 be the mean centres of the two partial groups for their respective systems of multiples, and P any arbitrary point not at infinity, as

$$\Sigma(a_1.A_1P^2)=\Sigma(a_1.A_1O_1^2)+\Sigma(a_1).O_1P^2,$$

and $$\Sigma(a_2.A_2P^2)=\Sigma(a_2.A_2O_2^2)+\Sigma(a_2).O_2P^2;$$

therefore, by addition, remembering that $\Sigma(a_1.A_1O_1^2)$ and $\Sigma(a_2.A_2O_2)^2$ are both constant, and that $\Sigma(a_1)+\Sigma(a_2)$ by hypothesis $=0$, it appears that the sum $\Sigma(a.AP^2)$ depends on the quantity $\Sigma(a_1).(O_1P^2-O_2P^2)$ or its equivalent $\Sigma(a_2).(O_2P^2-O_1P^2)$, that is, on the difference of the squares of O_1P and O_2P, and is therefore constant (Euc. I. 47, Cor.) when P is any where on the same line perpendicular to O_1O_2, and therefore &c. *In the exceptional case when O_1 and O_2 coincide, and when therefore O is indeterminate, as $O_1P^2-O_2P^2=0$ for every position of P not at infinity, the sum $\Sigma(a.AP^2)$ undergoes therefore no variation, but preserves in magnitude and sign the same constant value (which may $=0$) for all positions of P not at infinity.*

Cor. If I be the line bisecting at right angles the interval O_1O_2; since then (76), $O_1P^2-O_2P^2=2.O_1O_2.IP$, therefore $\Sigma(a_1).(O_1P^2-O_2P^2)$, or its equivalent $\Sigma(a_2).(O_2P^2-O_1P^2)$, $=2\Sigma(a_1).O_1O_2.IP$, or its equivalent $2\Sigma(a_2).O_2O_1.IP$, $=2k.IP$, (94); and therefore if P and Q be any two points on any two lines M and N parallel to I, that is, perpendicular to O_1O_2 the direction of O, then, by the above,

$$\Sigma(a.AP^2)-\Sigma(a.AQ^2)=2k.(IP-IQ)=2k.NM;$$

the same formula exactly as that found directly in the preceding and leading of course to the same consequences there given.

102. *If O be the centre of the circle inscribed in any triangle ABC, O', O'', O''' those of the three exscribed to the three sides a, b, c, and s the semi-perimeter, then*

1°. *For every arbitrary line L not at infinity,*

$$(s-a).O'L+(s-b).O''L+(s-c).O'''L-s.OL=0\ldots(1).$$

2°. *For every arbitrary point P not at infinity,*

$$(s-a).O'P^2+(s-b).O''P^2+(s-c).O'''P^2-s.OP^2=2abc\ldots(2).$$

To prove 1°. From the general relation $\Sigma(a.AL)=\Sigma(a).OL$, (92) applied successively to the four points O, O', O'', O''' regarded (91) as the four mean centres of the three points A, B, C for the four varieties of signs of the three multiples a, b, c,

$$\left.\begin{array}{l} a.AL+b.BL+c.CL=(a+b+c).OL \;=2s.OL \\ b.BL+c.CL-a.AL=(b+c-a).O'L \;=2(s-a).O'L \\ c.CL+a.AL-b.BL=(c+a-b).O''L=2(s-b).O''L \\ a.AL+b.BL-c.CL=(a+b-c).O'''L=2(s-c).O'''L \end{array}\right\}..(3),$$

and it is evident, from mere inspection of their right-hand numbers, that, as above stated, the first is = the sum of the other three.

To prove 2°. From the general relation

$$\Sigma(a.AP^2)-\Sigma(a.AO^2)=\Sigma(a).OP^2, \text{ (98)}$$

applied successively to the four points O, O', O'', O''' regarded as before, and remembering that by Cor. 1°. (99), $\Sigma(a.AO^2)=abc$, and that by the same $\Sigma(a.AO'^2)=\Sigma(a.AO''^2)=\Sigma(a.AO'''^2)=-abc$,

$$\left.\begin{array}{l} a.AP^2+b.BP^2+c.CP^2-abc=(a+b+c).OP^2 \;=2s.OP^2 \\ b.BP^2+c.CP^2-a.AP^2+abc=(b+c-a).O'P^2 \;=2(s-a).O'P^2 \\ c.CP^2+a.AP^2-b.BP^2+abc=(c+a-b).O''P^2=2(s-b).O''P^2 \\ a.AP^2+b.BP^2-c.CP^2+abc=(a+b-c).O'''P^2=2(s-c).O'''P^2 \end{array}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \text{ (4)},$$

the first of which subtracted from the sum of the other three, gives evidently the above relation multiplied by 2.

Cor. 1°. Since for every line L passing through any one of the four points O, O', O'', O''' the perpendicular from that point $=0$, therefore, by relation 1, *each of the four points* O, O', O'', O''' *is the mean centre of the remaining three for the corresponding three of the four multiples* $-s$, $s-a$, $s-b$, $s-c$; a property the reader may easily prove directly for himself.

Cor. 2°. Denoting by r, r', r'', r''' the radii of the four circles, and by R that of the circle circumscribing the triangle, it may be shown at once—

1°. From relation 1, that

$$\frac{O'L}{r'}+\frac{O''L}{r''}+\frac{O'''L}{r'''}-\frac{OL}{r}=0 \;\ldots\ldots\ldots\ldots \text{ (5)}.$$

2°. And from relation 2, that

$$\frac{O'P^2}{r'} + \frac{O''P^2}{r''} + \frac{O'''P^2}{r'''} - \frac{OP^2}{r} = 8R \text{ .... } (6),$$

for since by (92, Cor. 5°.) $(s-a)\,r' = (s-b)\,r'' = (s-c)\,r''' = sr =$ area of triangle $= \Delta$, and since by (64, Cor. 2°.) $abc = 4R\Delta$; therefore dividing 1 and 2 by Δ they assume at once the forms 5 and 6; from the first of which again, as in Cor. 1°. it follows that *each of the four points* O, O', O'', O''' *is the mean centre of the remaining three for the corresponding three of the four multiples* $-\frac{1}{r}, \frac{1}{r'}, \frac{1}{r''}, \frac{1}{r'''}$, the reciprocals of $-s$, $(s-a)$, $(s-b)$, $(s-c)$ *to the unit whose square* $= \Delta$; a property again as easily proved directly for the reciprocal as for the original multiples.

Cor. 3°. Conceiving in the four relations (4), the arbitrary point P to coincide with the centre of the circle circumscribing the triangle, and denoting in that case by D, D', D'', D''' the four distances OP, $O'P$, $O''P$, $O'''P$, then, as $AP = BP = CP = R$, the four relations become

$$\left.\begin{aligned} (a+b+c).R^2 - abc &= (a+b+c).D^2 \\ (b+c-a).R^2 + abc &= (b+c-a).D'^2 \\ (c+a-b).R^2 + abc &= (c+a-b).D''^2 \\ (a+b-c).R^2 + abc &= (a+b-c).D'''^2 \end{aligned}\right\} \text{ .... } (7),$$

which are the formulæ by which to calculate in numbers the four distances D, D', D'', D''' when the sides of the triangle are given; and from which again, as for any other position of P, it follows that

$$(s-a)\,D'^2 + (s-b)\,D''^2 + (s-c)\,D'''^2 - sD^2 = 2abc \text{ ... } (8).$$

Cor. 4°. Substituting in the four relations (7), for abc its value $4R\Delta$ (64, Cor. 2°.), and for s, $(s-a)$, $(s-b)$, $(s-c)$ their values $\frac{\Delta}{r}, \frac{\Delta}{r'}, \frac{\Delta}{r''}, \frac{\Delta}{r'''}$, we get at once the values of the four distances in the well known forms*

$$D^2 = R^2 - 2Rr,\quad D'^2 = R^2 + 2Rr',\quad D''^2 = R^2 + 2Rr'',$$
$$D'''^2 = R^2 + 2Rr''' \text{ .... } (9),$$

* See Galbraith and Haughton's *Manual of Euclid*, Book IV., Appendix.

from which it appears that *the radii of two circles and the distance between their centres must fulfil a certain relation of condition, in order to the possibility of a triangle being at once circumscribed to one of them and inscribed or exscribed to the other ;* a particular case of a more general property which will be given in another chapter.

Cor. 5°. If OT, $O'T'$, $O''T''$, $O'''T'''$ be the four tangents from the four points O, O', O'', O''' to the circle circumscribing the triangle; since then

$$OT^2 = D^2 - R^2, \quad O'T'^2 = D'^2 - R^2, \quad O''T''^2 = D''^2 - R^2, \quad O'''T'''^2 = D'''^2 - R^2,$$

therefore, by relations 7,

$$OT^2 = -\frac{abc}{a+b+c}, \quad O'T'^2 = \frac{abc}{b+c-a}, \quad O''T''^2 = \frac{abc}{c+a-b}, \quad O'''T'''^2 = \frac{abc}{a+b-c} \quad \ldots\ldots\ldots (10),$$

which are the formulæ by which to calculate in numbers the lengths of the four tangents OT, $O'T'$, $O''T''$, $O'''T'''$ when the sides of the triangle are given; and from which, as is otherwise evident, it appears that the first of the four OT is always imaginary and the remaining three $O'T'$, $O''T''$, $O'''T'''$ always real.

Cor. 6°. Taking the values of $D^2 - R^2$, $D'^2 - R^2$, $D''^2 - R^2$, $D'''^2 - R^2$ from relation 9, we see again that

$$OT^2 = -2Rr, \quad O'T'^2 = 2Rr', \quad O''T''^2 = 2Rr'', \quad O'''T'''^2 = 2Rr''' \quad \ldots\ldots\ldots (11),$$

which are the formulæ by which to calculate in numbers the length of any one of the four tangents OT, $O'T'$, $O''T''$, $O'''T'''$ when the radii of the circumscribed and of the corresponding inscribed or exscribed circles are given; and from which it follows at once

that $O'T'^2 : O''T''^2 : O'''T'''^2 : OT^2 = r' : r'' : r''' : -r \ldots (12)$,

that $\dfrac{1}{O'T'^2} + \dfrac{1}{O''T''^2} + \dfrac{1}{O'''T'''^2} + \dfrac{1}{OT^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$,

that $O'T'^2 + O''T''^2 + O'''T'''^2 + OT^2 = 8R^2 \quad \ldots\ldots\ldots\ldots\ldots (14)$,

and that each of the four points O, O', O'', O''' is the mean centre of the remaining three for the corresponding three of the four multiples $\frac{1}{OT^2}$, $\frac{1}{O'T'^2}$, $\frac{1}{O''T''^2}$, $\frac{1}{O'''T'''^2}$.

COR. 7°. Regarding the point O as the mean centre of the three O', O'', O''' for the three multiples $\frac{1}{r'}$, $\frac{1}{r''}$, $\frac{1}{r'''}$, (Cor. 2°.) it follows at once from the general relation

$$\Sigma\,(ab\,.\,AB^2) = \Sigma\,(a)\,.\,\Sigma\,(a\,.\,AO^2)\ (99),$$

that

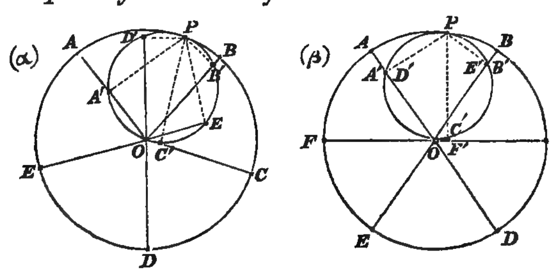
$$\frac{O''O'''^2}{r''r'''} + \frac{O'''O'^2}{r'''r'} + \frac{O'O''^2}{r'r''} = \frac{OO'^2}{rr'} + \frac{OO''^2}{rr''} + \frac{OO'''^2}{rr'''} = 8\,\frac{R}{r}\ \ldots\ (15),$$

which would also follow at once, as in the general relation referred to, by conceiving the arbitrary point P in relation 6, Cor. 2°., to coincide successively with each of the four points O, O', O'', O'''.

103. *If P be any point on the circle passing through the several vertices A, B, C, D, &c. of a regular polygon of any order n, and L any line passing through the centre O of the figure, then—*

1°. *The sum of the squares of the perpendiculars from P upon the several radii OA, OB, OC, OD, &c. is constant, and $= \frac{1}{2}n$ times the square of the radius of the circle.*

2°. *The sum of the squares of the perpendiculars upon L from the several vertices A, B, C, D, &c. is also constant, and $= \frac{1}{2}n$ times the square of the radius of the circle.*



To prove 1°. On the radius OP as diameter conceiving another circle described intersecting the several radii OA, OB, OC, OD, &c. in the feet A', B', C', D', &c. of the several perpendiculars upon them from P; then the several angles $A'OB'$,

$B'OC'$, $C'OD'$, $D'OE'$, &c. being equal, and each = the n^{th} part of four right angles, the several points A', B', C', D', &c. form therefore on the auxiliary circle, if n be odd (fig. α) the n vertices of a regular polygon of the order n, and if n be even (fig. β) the $2\,\frac{n}{2}$ vertices of two coincident regular polygons of the order $\frac{n}{2}$ (since in that case they evidently coincide two and two in opposite pairs), and therefore in either case $\Sigma(PA'^2)$ (98), Cor. 4°. = $2n$ times the square of the radius of the auxiliary, that is = $\frac{1}{2}n$ times the square of the radius of the original circle, and therefore &c.

N.B. In the same way exactly, it appears that the sum of the squares of the several intercepts OA', OB', OC', OD', &c. between the centre of the circle and the feet of the several perpendiculars from P upon the n radii OA, OB, OC, OD, &c. is constant, and = $\frac{1}{2}n$ times the square of the radius of the circle.

To prove 2°. Since for any two points on a circle, the perpendicular from either upon the diameter passing through the other = the perpendicular from the latter upon the diameter passing through the former, therefore the several perpendiculars from the n points A, B, C, D, &c. upon the one diameter passing through any other point P on the circle = the several perpendiculars from the one point P upon the several diameters passing through the n points A, B, C, D, &c.; but by 1°. the sum of the squares of the latter is constant, and = $\frac{1}{2}n$ times the square of the radius of the circle, therefore so is also the sum of the squares of the former, and therefore &c.

N.B. In the same way exactly it appears, from the note to 1°., that the sum of the squares of the several intercepts between the centre of the circle and the feet of the several perpendiculars from the n vertices A, B, C, D, &c. upon any diameter L is constant, and = $\frac{1}{2}n$ times the square of the radius of the circle.

Cor. 1°. From the above 1°. and 2°. combined with the two properties of regular polygons given in (92, Cor. 6°.) it follows that—

If O be the centre of a regular polygon of any order n, OQ

and OR the radii of its inscribed and circumscribed circles, P any arbitrary point, and L any arbitrary line, then—

1°. *The sum of the squares of the perpendiculars from P upon the several sides is constant and variable with OP, and*

$$= n\,(OQ^2 + \tfrac{1}{2}OP^2).$$

2°. *The sum of the squares of the perpendiculars upon L from the several vertices is constant and variable with OL, and*

$$= n\,(OL^2 + \tfrac{1}{2}OR^2).$$

To prove 1°. Since by (92, Cor. 6°.), the sum of the perpendiculars from P on the several sides $= n$ times OQ, therefore by (79, Cor. 2°.) the sum of their squares $= n$ times $OQ^2 +$ the sum of the squares of the n differences between each of themselves and OQ; but the circle on OP as diameter intersecting the several perpendiculars from P in the feet of the several perpendiculars upon them from O, and intercepting therefore upon them the several differences in question, therefore by the above 1°. the sum of the squares of the n differences $= \frac{1}{2}n$ times the square of OP, and therefore &c.

To prove 2°. Since by (92, Cor. 6°.) the sum of the perpendiculars upon L from the several vertices $= n$ times OL, therefore by (79, Cor. 2°.) the sum of their squares $= n$ term $OL^2 +$ the sum of the squares of the n differences between each of themselves and OL; but the parallel to L passing through O cutting off from the several perpendiculars the n differences in question, therefore by the above 2°. the sum of the squares of the n differences $= \frac{1}{2}n$ times the square of OP, and therefore &c.; and the same thing is also evident from the general property (96, Cor. 1°.) of which this is evidently a particular case.

N.B. It is evident from the above 1°. and 2°. that every circle concentric with a regular polygon of any order, is at once the locus of a variable point the sum of the squares of whose distances from the several sides is constant, and the envelope of a variable line the sum of the squares of whose distances from the several vertices is constant.

Cor. 2°. Conceiving, in the above, the arbitrary point P to be on the circle OQ, and the arbitrary line L to touch the circle OR, it follows at once that—

1°. *If from any point on the circle inscribed in a regular polygon of any order n perpendiculars be let fall on the several sides, the sum of their squares is constant, and* $= \frac{3}{2}n.\text{radius}^2$ *of circle.*

2°. *If upon any line L touching the circle circumscribed to a regular polygon of any order n perpendiculars be let fall from the several vertices, the sum of their squares is constant, and* $= \frac{3}{2}n.\text{radius}^2$ *of circle.*

For, when in 1°. $OP = OQ$, then $OQ^2 + \frac{1}{2}OP^2 = \frac{3}{2}OQ^2$, and therefore &c.; and when in 2°. $OL = OR$, then $OL^2 + \frac{1}{2}OR^2 = \frac{3}{2}OR^2$, and therefore &c.

Cor. 3°. Comparing with each other the values of the two sums of squares in the particular cases just given, it follows also that—

1°. *If two regular polygons of any common order n be constructed one circumscribed and the other inscribed to the same circle, the constant sum of the squares of the perpendiculars from any point on the circle upon all the sides of the former = the constant sum of the squares of the perpendiculars upon any tangent to the circle from all the vertices of the latter.*

2°. *If two circles be described one circumscribed and the other inscribed to the same regular polygon of any order n, the constant sum of the squares of the perpendiculars upon all the sides of the polygon from any point on the former = the constant sum of the squares of the perpendiculars from all the vertices of the polygon upon any tangent to the latter.*

For, by the above 1°. and 2°. both constant sums, in the former case $= \frac{3}{2}n.\text{radius}^2$ of common circle, and in the latter case $= n.\text{radius}^2$ of inscribed circle $+ \frac{1}{2}n.\text{radius}^2$ of circumscribed circle, and therefore &c.

CHAPTER VII.

ON COMPLETE AND INCOMPLETE FIGURES OF POINTS AND LINES.

104. EVERY system of points or lines, whatever be their number and position, in which the several lines of connection or points of intersection of each point or line with all the others are taken into account without exception, is said to form a *complete* figure, which in the absence of any as yet generally recognized nomenclature may be termed a *polystigm* in the former case and a *polygram* in the latter. A system of points or lines, on the other hand, in which any of the lines of connection or points of intersection of each point or line with all the others are omitted, is said to form an *incomplete* figure, whose degree of incompleteness depends of course on the number of the omitted points or lines. In the extreme case of the latter, when the lines of connection or points of intersection of each point or line with but two others are taken into account, the figure evidently is simply a *polygon*, of which the several points or lines of the system are the several vertices or sides, and of which the shape and character depend, of course, on the *order of sequence* in which the several points or lines of the system are taken in the several connections or intersections of each with the two regarded as adjacent to it.

105. The several points or lines constituting the vertices or sides of a polygon of any order being always taken in some definite order of sequence, it is therefore an intelligible mode of expression to speak, as is often done, of "opposite vertices" and of "opposite sides" in one of an even order, or, of "the vertex opposite to a side" and of "the side opposite to a vertex" in one of an odd order; but to speak similarly of the constituent points or lines determining a complete figure of any

order would be meaningless and consequently inadmissible; each point or line standing by itself absolutely, and having no relation of the nature expressed by such terms as "adjacent," "opposite," &c. to any other.

106. But though *the determining points or lines* in complete figures have no relation amongst each other as regards order of sequence, *certain other elements* of the figures may be, and often are, with propriety and convenience, said to be *opposites* to each other; thus, for instance, in a tetrastigm or tetragram every line of connection of two points or point of intersection of two lines is said to be the opposite of that of the remaining two; in a hexastigm or hexagram every triangle determined by three points or lines is said to be the opposite of that determined by the remaining three; and, generally in a polystigm or polygram of any even order, every two polystigms or polygrams of inferior orders determined by half the points or lines and by the remaining half are said to be opposites to each other, &c.

107. In the complete figure determined by any number of points, every two points are said to determine *a line of connection*, and every two lines of connection to determine *an angle of connection* of the figure. In the complete figure determined by any number of lines, every two lines are said to determine *a point of intersection*, and every two points of intersection to determine *a chord of intersection* of the figure; for the same obvious reason as for the extreme case of incomplete figures, the several chords of intersection in the latter case are sometimes termed also *diagonals* of the figure.

108. If n be the number of the points or lines determining a complete figure of either species, it may be easily shown that, generally:

1°. *The entire number of lines of connection or of points of intersection of the figure* $= \dfrac{n(n-1)}{2}$.

2°. *The entire number of angles of connection or of chords of intersection of the figure* $= \dfrac{n(n-1)(n-2)(n-3)}{8}$.

3°. *The entire number of polygons of which the determining points or lines are the vertices or sides* $= \frac{(n-1)(n-2)(n-3)\ldots 1}{2}$.

For, in the case of 1°., the n points or lines connecting or intersecting each with the remaining $(n-1)$ produce $n(n-1)$ lines of connection or points of intersection *coinciding in pairs*, and therefore &c.; in the case of 2°., the $\frac{n(n-1)}{2}$ lines of connection or points of intersection of two points or lines intersecting or connecting with the $\frac{(n-2)(n-3)}{2}$ for the remaining $(n-2)$ produce $\frac{n(n-1)(n-2)(n-3)}{4}$ angles of connection or chords of intersection *coinciding in pairs*, and therefore &c.; and, in the case of 3°., any one of the n points or lines, taken arbitrarily as *first* vertex or side of all the polygons, may be followed in order of sequence by any one of the remaining $(n-1)$ as *second* vertex or side, each of which $(n-1)$ second vertices or sides may be followed in order of sequence by any one of the remaining $(n-2)$ as *third* vertex or side, each of which $(n-1)(n-2)$ third vertices or sides may be followed in order of sequence by any one of the remaining $(n-3)$ as *fourth* vertex or side, and so on to the last, thus producing $(n-1).(n-2).(n-3).(n-4)$, &c. 2.3.1 *last* vertices or sides, and therefore the same number of polygons *coinciding in pairs*, every order of sequence giving evidently the same polygon as the reverse order, and therefore &c.

109. A polygon of any order greater than three is said to be *convex*, *reentrant*, or *intersecting*, according as every two of its non-adjacent sides intersect externally, as any two of them intersect one externally and one internally, or as any two of them intersect internally; thus the quadrilateral $ABCD$ in

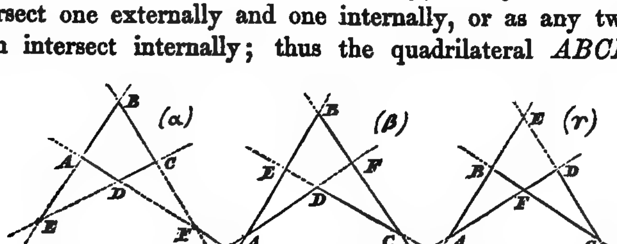
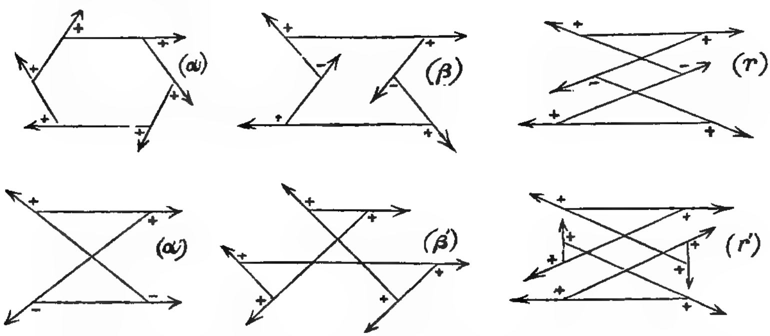


fig. α is convex, in fig. β is reentrant, and in fig. γ is intersecting,

and it is evident from any of the three figures that of the three different quadrilaterals $ABCD$, $AECF$, $BEDF$, determined by the same four lines (108), one is always convex, one always reentrant, and one always intersecting.

110. *The sum of the external angles of a polygon of any order, convex, reentrant, or intersecting, regard being had to their signs as well as their magnitudes,* $= \pm 4m$ *right angles,* m *being some integer of the natural series* 0, 1, 2, 3, *&c. less than half the order of the polygon.*



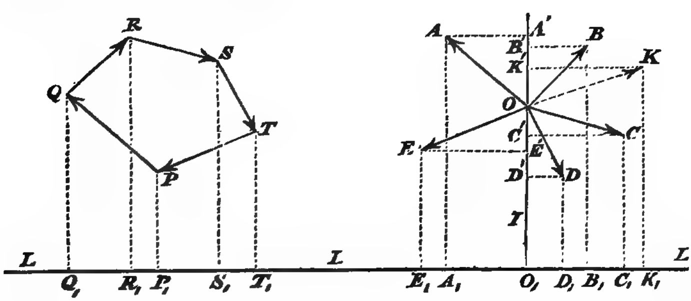
For, conceiving the polygon described by the motion of a point setting out from any position on one of its sides, and traversing its entire perimeter, returning again to the point of starting; the several *external* angles of the polygon are then evidently the several *deviations to the right or left*, in the direction of its motion, made by the describing point in passing during the circuit from the several sides to their successors, which for convex polygons universally (fig. α), and for others too occasionally (figs. β' and γ'), take place all in the same direction, and have therefore all the same sign, but which for reentrant and intersecting polygons generally (figs. β, γ, and α') take place some in one and others in the opposite direction, and have therefore some one and others the opposite sign; but since, on the completion of the entire circuit, the original direction of the motion is always finally regained, therefore *the total amount of deviation* however made up, that is the sum with their proper signs of the external angles of the polygon, $= 0$, or $= 4$ right angles, or $= 4m$ right angles, m however being

always less than half the order of the polygon, the deviation at each angle being necessarily limited to two right angles.

In the three polygons represented in figs. α, β, and γ, and in all convex polygons universally, $m=1$; in the three represented in figs. α', β', and γ', $m=0, =2$, and $=3$ respectively; and in all six alike the sides are supposed to be described in the directions indicated by the arrow heads in the figures, and the deviations are supposed to be positive or negative according as they take place to the right or to the left, as marked in the figures.

Any two sides of a polygon are said *to be measured cyclically in similar or opposite directions*, according as a moving point, going round as above the entire perimeter continuously in the same cyclic direction, would describe both in directions similar or opposite to those of their measurement or describe one in the similar and the other in the opposite direction.

111. *If the several sides of any polygon measured cyclically in the same direction be projected in any direction upon any line, the sum of the projections, regard being had to their signs as well as to their magnitudes*, $=0$.



For, if P, Q, R, S, T, &c. be the several vertices of the polygon, and P_1, Q_1, R_1, S_1, T_1, &c. their several projections upon any arbitrary line L, then the several sides, measured cyclically in the common direction indicated by the arrow heads in the figure, being PQ, QR, RS, ST, &c. returning back again to P, their several projections on L are respectively P_1Q_1, Q_1R_1, R_1S_1, S_1T_1, &c. returning back again to P_1, and the sum of the latter (by 78) being always $=0$, therefore &c.

The above useful property may obviously be stated otherwise thus, as follows—

If the several sides of any polygon be projected in any direction upon any line, the projection of any one side measured cyclically in either direction, or more generally the sum of the projections of any number of the sides measured cyclically in either direction, is equal to the sum of the projections of the remaining sides measured cyclically in the opposite direction.

112. Assuming the evident property that, if two finite parallel lines, however circumstanced as to absolute position, be equal and co-directional, their projections in any direction upon any line are equal and co-directional; the following consequences result immediately from the very useful property of the preceding article, viz.—

1°. *A system of any number of finite lines given in length and direction but not in absolute position would form a polygon if placed end to end in any order of sequence, provided that for two different directions of projection the sum of their projections upon any line* $=0$.

For, if when placed end to end in any one of the different orders of sequence in which they could be disposed, the last extremity of the last side failed to coincide with the first extremity of the first side, then, though the sum of their projections would $=0$ upon every line *for the particular direction of projection parallel to the line connecting those two extremities*, such obviously would not be the case upon any line for *any other* direction of projection, and therefore &c.

2°. *If a system of any number of finite lines given in length and direction but not in absolute position would form a polygon if placed end to end in any one order of sequence, they would do so equally for every order of sequence in which they could be disposed.*

For, if for any one order of sequence they formed a polygon, then since, by (111), the sum of their projections in every direction upon every line $=0$, therefore, by 1°., they would form a polygon for every order of sequence, and therefore &c.

3°. *If a system of any number of finite lines, however circumstanced as to direction, length, and position, be such that for two*

different directions of projection the sum of their projections upon any line $=0$, *then for every direction of projection the sum of their projections upon any line* $=0$.

For, if without alteration of length or direction they were, if not already in such a position, placed end to end in any order of sequence, then, since by hypothesis the sum of their projections for two different directions $=0$, therefore, by 1°., they would form a polygon, and therefore, by (111), the sum of their projections for every direction $=0$.

113. If from any point O as common origin (fig., Art. 111) a system of finite lines OA, OB, OC, OD, &c. be drawn parallel, equal, and co-directional to the several sides PQ, QR, RS, ST, &c. of any polygon $PQRST$ &c. measured cyclically in the same direction, it is easy to see from the same property that—

1°. *The sum of their projections in any direction upon any line* $=0$.

2°. *The sum of the perpendiculars, or any other isoclinals, from their extremities upon any line passing through* $O=0$.

3°. *The sum of the areas of the triangles they subtend at any point not at infinity* $=0$.

4°. *The sum of the rectangles under them and the perpendiculars upon them from any point not at infinity* $=0$.

To prove 1°. and 2°. If O_1, A_1, B_1, C_1, D_1, &c. be the several projections in any direction OO_1 of the several points O, A, B, C, D, &c. upon any line L; P_1, Q_1, R_1, S_1, T_1, &c. those of P, Q, R, S, T, &c. in the same direction on the same line, and AA', BB', CC', DD', &c. the several isoclinals from A, B, C, D, &c. to OO_1 in the direction parallel to L; then since, by hypothesis, Art. 112, and Euc. I. 34, $P_1Q_1=O_1A_1=A'A$, $Q_1R_1=O_1B_1=B'B$, $R_1S_1=O_1C_1=C'C$, $S_1T_1=O_1D_1=D'D$, &c., and since, by (111),

$$P_1Q_1+Q_1R_1+R_1S_1+S_1T_1+\&c.=0,$$

therefore $$O_1A_1+O_1B_1+O_1C_1+O_1D_1+\&c.=0,$$

and $$AA'+BB'+CC'+DD'+\&c.=0,$$

and therefore &c., the directions of L and of OO_1 being entirely arbitrary.

To prove 3°. If I be the point and AA', BB', CC', DD', &c. the several perpendiculars from A, B, C, D, &c. upon

the line OI passing through the two points O and I, then since, by 2°.,

$$AA' + BB' + CC' + DD' + \&c. = 0,$$

and since by hypothesis OI is not infinite, therefore

$$OI.AA' + OI.BB' + OI.CC' + OI.DD' + \&c. = 0,$$

and therefore &c., each rectangle being double the area of the triangle subtended by its base at the point I.

To prove 4°. If I, as before, be the point and IX, IY, IZ, &c. the several perpendiculars from it upon OA, OB, OC, &c., then since $OA.IX = 2$ area OAI, $OB.IY = 2$ area OBI, $OC.IZ = 2$ area OCI, &c., and since, by 3°.,

$$2 \text{ area } OAI + 2 \text{ area } OBI + 2 \text{ area } OCI + \&c. = 0,$$

therefore $\quad OA.IX + OB.IY + OC.IZ + \&c. = 0,$

and therefore &c.

Of the above properties, 2°. shews evidently (86) that the point O is the mean centre of the system of points A, B, C, D, &c. for any system of multiples having a common magnitude and sign; and 4°. expresses obviously for any number of lines OA, OB, OC, OD, &c. passing through a common point O, what the property, Cor. 6°., Art. 82, established on other considerations in Chapter V., expresses for three.

114. When any number of lines OA, OB, OC, OD, &c. diverging from a common origin O, are, as in the preceding article, parallel, equal, and co-directional to the several sides of a polygon $PQRST$ &c. measured cyclically in the same direction, any one of them OE turned without change of length round the common origin O into the opposite direction OK is termed *the resultant* of the others OA, OB, OC, OD, &c., a name borrowed from the Science of Mechanics, in which the properties of the preceding and of some of the following articles are of considerable importance.

As all the sides but one of a polygon of any order are of course perfectly arbitrary in length and direction, the length and direction of the last however being implicitly given with those of the others, therefore the several lines OA, OB, OC, OD, &c. composing the system of which OK is the resultant

as above defined are equally arbitrary in length and direction, but their lengths and directions once given their resultant in length and direction is implicitly given with them; two very rapid constructions for its determination in all cases will be presently given.

In the particular case of but two components OA and OB, the resultant OK in length and direction is evidently the conterminous diagonal of the parallelogram of which OA and OB in length and direction are adjacent sides. *All properties therefore which are true in general of any system of coinitial lines and their resultant are true in particular of two adjacent sides and the conterminous diagonal of any parallelogram.*

115. Since, in accordance with the foregoing definition, the several pairs of magnitudes OE and OK, OE' and OK', OEI and OKI, O_1E_1 and O_1K_1, EE' and KK', &c., in the figure of Art. 111, are equal and opposite, it follows at once from the several properties of Article 113 that the resultant OK of any system of lines OA, OB, OC, OD, &c. diverging from a common origin O possesses the following properties with respect to the component lines of the system—

1°. *The sum of the projections of the components in any direction upon any line is equal in magnitude and sign to the projection of the resultant in the same direction upon the same line.*

2°. *The sum of the perpendiculars or other isoclinals from the extremities of the components upon any line passing through the common origin O is equal in magnitude and sign to the perpendicular or isoclinal from the extremity of the resultant on the same line.*

3°. *The sum of the areas of the triangles subtended by the components at any point not at infinity is equal in magnitude and sign to the area of the triangle subtended by the resultant at the same point.*

4°. *The sum of the rectangles under the components and the perpendiculars upon them from any point not at infinity is equal in magnitude and sign to the rectangle under the resultant and the perpendicular upon it from the same point.*

These properties require no proof, they result immediately, as above enumerated from those similarly mentioned in Art. 113,

from the obvious consideration that when the sum of a number of magnitudes of any kind $= 0$ then any one of them changed in sign $=$ the sum of all the others; and it follows at once from any or all of them, as is also evident from the fundamental definition of the preceding article, that for a system of components parallel, equal, and co-directional to the several sides of any polygon measured cyclically in the same direction, the resultant is in magnitude evanescent and in direction indeterminate.

116. *Given in magnitude and direction any number of lines OA, OB, OC, OD, &c. diverging from a common origin O, to determine their resultant OK in magnitude and direction.*

First method. From any arbitrarily assumed point P (fig., Art. 111), drawing a line PQ parallel, equal, and co-directional to any one of the components OA; from its opposite extremity Q a second QR parallel, equal, and co-directional to a second of them OB; from its opposite extremity R a third RS parallel, equal, and co-directional to a third of them OC; from its opposite extremity S a fourth ST parallel, equal, and co-directional to a fourth of them OD; and so on until all the components are exhausted. The line OK from O parallel, equal, and co-directional to the line PT connecting the first extremity P of the first parallel PQ with the last extremity T of the last parallel ST is (114) the resultant required.

Should the last point T, determined by this construction, coincide with the first point P, assumed originally, that is, should the given lines OA, OB, OC, OD, &c. form a system parallel, equal, and co-directional to the several sides of any polygon measured cyclically in the same direction; their resultant OK, thus determined would, as it ought (115), be evanescent in magnitude and indeterminate in direction.

Second method. Projecting all the components OA, OB, OC, OD, &c. in any direction upon any line OO_1 (same figure) passing through their common origin O, and measuring from O on OO_1 a length OK' equal in magnitude and sign to the sum of the several projections OA', OB', OC', OD', &c., the length OK' thus determined is (115) the corresponding projection of the required resultant OK. Repeating the same process with a different direction of projection on the same or another line

passing through O, the new length similarly determined is a second projection of the required resultant OK, and therefore &c.

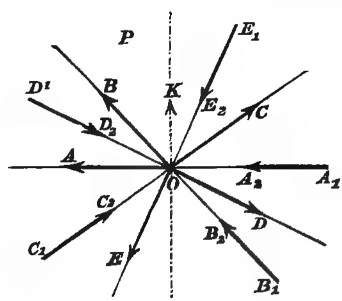
Should the two different lengths, determined as above, be both $=0$, that is, (112, 1°.) should the given lines OA, OB, OC, OD, &c. form a system parallel, equal, and co-directional to the several sides of any polygon measured cyclically in the same direction, their resultant OK thus determined would again, as it ought, be evanescent in magnitude and indeterminate in direction.

Of the above two general constructions the second, though less obvious and simple, is better adapted to numerical computation than the first.

117. The principles established in the preceding articles supply a ready solution of the very general problem—

Required the locus of a variable point P for which the sum of the areas of the system of triangles A_1PA_2, B_1PB_2, C_1PC_2, D_1PD_2, &c., subtended by any number of fixed bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. is constant, the length and line of direction with the positive and negative sides of each base being given.

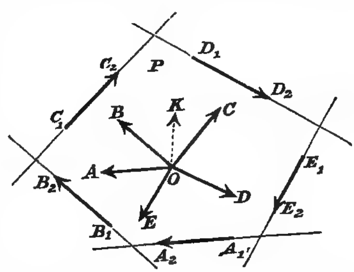
Case 1°. When the several lines of direction of the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. pass through a common point O; from the common point O measuring on the several lines of direction lengths OA, OB, OC, OD, &c., equal to the several lengths A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., and in directions, indicated by the arrow heads in the figure, such that the positive and negative sides of the several bases correspond to the right and left sides of the several directions, and taking, by (116), in length and direction the resultant OK of the several coinitial lines OA, OB, OC, OD, &c. thus obtained; then for every arbitrary point P not at infinity, since, by Euc. I. 38, the sum of the system of triangles $\Sigma(A_1PA_2)$ = the sum of the system of triangles $\Sigma(OPA)$, and since, by (115, 3°.), the sum of the latter system of triangles = the single triangle OPK, therefore



for every position of P not at infinity the sum of the system of triangles $\Sigma(A_1PA_2)$ = the single triangle OPK; but the base OK of the latter being fixed its area is positive, negative, or nothing, according as its vertex P lies on the right or left side of or upon the line of direction of OK, and if its area is constant the locus of its vertex P is a line parallel to OK and at a distance from it equal in magnitude and sign to twice the constant value divided by OK.

Cor. In the particular case when $OK=0$, that is, when the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. are parallel, equal, and co-directional with the several sides of a polygon measured cyclically in the same direction; then, as is evident from the above, *the sum of the areas of the system of triangles* $\Sigma(A_1PA_2)=0$ *for every position of P not at infinity.*

Case 2°. When the several lines of direction of the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. do not pass through a common point; assuming arbitrarily any fixed point O not at infinity, drawing from it a system of lines OA, OB, OC, OD, &c., parallel and equal to the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., and in directions, indicated



by the arrow heads in the figure, such that the positive and negative sides of the several bases correspond as before to the right and left sides of the several directions, and taking as before in magnitude and direction the resultant OK of the system of coinitial lines OA, OB, OC, OD, &c. thus obtained; then for every arbitrary point P not at infinity, since, by (75), the sum of the system of triangles $\Sigma(A_1PA_2)$ = the sum of the system of triangles $\Sigma(A_1OA_2)$ + the sum of the system of triangles $\Sigma(OPA)$, and since, by (115, 3°.), the sum of the latter system of triangles = the single triangle OPK, therefore for every position of P not at infinity, the sum of the system of triangles $\Sigma(A_1PA_2)$ = the sum of the system of triangles $\Sigma(A_1OA_2)$ + the single triangle OPK; but the sum of the system of triangles $\Sigma(A_1OA_2)$ being fixed with the point O, and the base OK of

the single triangle OPK being also fixed with the same, if the sum of the system of triangles $\Sigma(A_1PA_2)$ be constant, the locus of P is a line parallel to OK and distant from it by an interval equal in magnitude and sign to the constant sum $\Sigma(A_1PA_2)$ – the fixed sum $\Sigma(A_1OA_2)$ divided by half the length of OK.

Cor. In the particular case when $OK = 0$, that is, when the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. are parallel, equal, and co-directional with the several sides of a polygon measured cyclically in the same direction, then, as is evident from the above, *the sum of the areas of the system of triangles $\Sigma(A_1PA_2)$ is constant for every position of P not at infinity.*

118. As the several fixed bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., in the general case of the preceding, may be in length and direction the several sides of any one of the different polygons determined by their several lines of direction (108, 3°.) measured cyclically in the same direction, and as then, by the corollary to that case, the sum of the areas of the several triangles A_1PA_2, B_1PB_2, C_1PC_2, D_1PD_2, &c. is constant for every position of P not at infinity; hence the important property that—

For a polygon of any form, convex, reentrant, or intersecting, the sum of the several triangular areas subtended by the several sides at any point not at infinity is constant, any two of the triangles being regarded as having similar or opposite signs according as they lie at similar or opposite sides of their respective bases measured cyclically in either common direction.

This property is important as supplying a formal *definition* of the *area* of a *polygon*, which is applicable without exception to every variety of form whether convex, reentrant, or intersecting, viz., "The constant sum of the areas of the several triangles subtended by the several sides at any arbitrary point not at infinity and regarded as positive or negative according as they lie at the positive or negative sides of their several bases measured cyclically in either common direction."

If an intersecting polygon were of such a form that the sum of the triangular elements constituting its area as thus defined $= 0$ for any one point not at infinity, they would of course by virtue of the above $= 0$ for every point not at infinity, and the area of the polygon would consequently $= 0$; an inter-

secting quadrilateral in which the two opposite sides that do not intersect internally are equal and parallel, (as in fig. α', Art. 110), furnishes the simplest example of a polygon of this nature.

119. The linear locus in the general case of Art. 117 supplies obvious solutions of the four following very general problems—

Given in magnitude, position, and direction any number of fixed bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, *&c. to determine*—

1°. *On a given line the point* P *for which the sum of the several triangular areas* $\Sigma(A_1PA_2)$ *shall have any given value, positive, negative, evanescent, or infinite.*

2°. *On a given circle the point* P *for which the sum of the several triangular areas* $\Sigma(A_1PA_2)$ *shall have the maximum, the minimum, or any intermediate given value.*

In the particular case when the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. are parallel, equal, and co-directional with the several sides of a polygon measured cyclically in the same direction, the sum $\Sigma(A_1PA_2)$ being then constant for every position of P not at infinity, these several problems are in consequence all indeterminate.

120. Denoting by A, B, C, D, &c. the several indefinite lines of direction, and by a, b, c, d, &c. the several numerical values to any unit of linear measure of the several fixed bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. in the linear locus of Art. 117; it follows immediately from the general property there established that—

If A, B, C, D, *&c. be any system of lines disposed in any manner, but none infinitely distant, and* a, b, c, d, *&c. any system of corresponding multiples positive or negative, but none infinitely great, the locus of a variable point* P *for which the sum*

$$a.PA+b.PB+c.PC+d.PD+\&c.,$$

or more shortly $\Sigma(a.PA)$, *has any constant value, positive, negative, or nothing, is a line whose direction depends on the directions of the lines and the ratios of the multiples, and whose position depends on the value of the constant.*

The positions and sides, positive and negative, of the several lines A, B, C, D, &c., and the magnitudes and signs, positive or negative, of the several multiples a, b, c, d, &c. being given,

to determine the common direction of the several loci for all values of the constant, the particular position of the locus for any particular value of the constant, and the law governing the variation of the locus for the variation of the constant; on the several lines A, B, C, D, &c. from any arbitrarily assumed points A_1, B_1, C_1, D_1, &c., taking any system of lengths A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., proportional to the numerical values of the several multiples a, b, c, d, &c., and in directions, indicated by the arrow heads in the figures, such that the positive and negative signs of the several products $a.PA$, $b.PB$, $c.PC$, $d.PD$, &c. shall correspond to the right and left sides of the several directions; then since for every position of P not at infinity $A_1A_2.PA = 2$ area A_1PA_2, $B_1B_2.PB = 2$ area B_1PB_2, $C_1C_2.PC = 2$ area C_1PC_2, $D_1D_2.PD = 2$ area D_1PD_2, &c., and since therefore $\Sigma(A_1A_2.PA) = 2\Sigma(A_1PA_2)$, therefore, by (117), the locus of P for which $\Sigma(A_1A_2.PA)$ has any constant value, positive, negative, or nothing, is a line L parallel to the resultant OK of any coinitial system of lines OA, OB, OC, OD, &c., parallel, equal, and co-directional with A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., and distant from it by an interval equal in magnitude and sign to the quantity $\dfrac{\Sigma(A_1A_2.PA) - \Sigma(A_1A_2.OA)}{OK}$, or to its equivalent $\dfrac{\Sigma(a.PA) - \Sigma(a.OA)}{k}$, where k is the numerical value of OK to the same unit that a, b, c, d, &c. are those of OA, OB, OC, OD, &c.

If I be the particular line of the system parallel to OK for which the value of the constant $= 0$, it is easy to see that for any other line L of the system its value $= k.LI$; for, since for any two points P and Q on any two lines L and M parallel to OK, by (117),

$\Sigma(a.PA) = \Sigma(a.OA) + k.LO$ and $\Sigma(a.QA) = \Sigma(a.OA) + k.MO$, therefore at once, by subtraction,

$$\Sigma(a.PA) - \Sigma(a.QA) = k.LM,$$

and therefore if M be the particular line I of the system for every point Q of which $\Sigma(a.QA) = 0$, then for every point P of any other line L of the system $\Sigma(a.PA) = k.LI$, as above stated.

Given the particulars of the system of lines A, B, C, D, &c. and of the system of multiples a, b, c, d, &c. to determine the line I. Assuming arbitrarily any point O, and drawing from it in magnitude and direction the resultant OK of the coinitial system of lines parallel, equal, and co-directional with the several segments A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. determined as above, the line I parallel to OK, distant from it by the interval $\Sigma(a.OA) \div k$, and at the positive or negative side of its direction according as the sign of $\Sigma(a.OA)$ is negative or positive, is, by the above, that required.

The line I, for every point Q of which the constant sum $\Sigma(a.QA) = 0$, is termed *the central axis* of the system of lines A, B, C, D, &c. for the system of multiples a, b, c, d, &c., and, by aid of it, determined as above or otherwise, the position of the parallel line L for every point P of which the constant sum $\Sigma(a.PA)$ shall have any given value, positive or negative, is given at once by the above; for it is distant from I by the interval $\Sigma(a.PA) \div k$, and it lies at its positive or negative side according as the sign of $\Sigma(a.PA)$ is positive or negative.

In the particular case when $k = 0$, that is (116), when the several segments A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., determined as above, are parallel, equal, and co-directional with the several sides of a polygon measured cyclically in the same direction, *the central axis I is at infinity, except only when the value of $\Sigma(a.PA)$, which* (117, Cor.) *is then constant for every position of P not at infinity, $= 0$, in which exceptional case it is indeterminate.* And for the same reason generally the several parallel loci of the present article are all at infinity, except only the particular one corresponding to the constant value of $\Sigma(a.PA)$, which one is indeterminate.

121. *If A, B, C be any three lines, I their central axis for any three multiples a, b, c, and P, Q, R the three points at which A, B, C intersect with I, then always* (see 91, 1°.)

$$b.PB + c.PC = 0,\quad c.QC + a.QA = 0,\quad a.RA + b.RB = 0.$$

For, since for every three points P, Q, R on I, by the preceding, $a.PA + b.PB + c.PC = 0$, $a.QA + b.QB + c.QC = 0$, $a.RA + b.RB + c.RC = 0$; if P lie on A, then $b.PB + c.PC = 0$;

if Q lie on B, then $c.QC+a.QA=0$; if R lie on C, then $a.RA+b.RB=0$; and therefore &c.

Of the above, which supplies an obvious and very rapid method of determining the central axis I of any three lines A, B, C for any three multiples a, b, c, the two following particular cases are deserving of attention. See (91, Cor.).

1°. If in absolute magnitude $a=b=c$ the three lines connecting P, Q, R with the three opposite vertices bisect (61) the three opposite angles BC, CA, AB of the triangle ABC, all externally, or one externally and two internally, according as the signs of a, b, c are all similar, or that of one opposite to those of the other two.

2°. If in absolute magnitude $a:b:c$ as the lengths of the three corresponding sides of the triangle ABC, the three points P, Q, R bisect (65, Cor. 3°.) the three sides on which they lie, all externally, or one externally and two internally, according as the signs of a, b, c are all similar, or that of one opposite to those of the other two.

In the first case of 2°., the three points of external bisection of the three sides of the triangle ABC being at infinity, so therefore is the central axis I which contains them; this is in exact accordance with the closing observation of the preceding article, the three segments A_1A_2, B_1B_2, C_1C_2, determined as there directed on the three lines A, B, C, being then parallel, equal, and co-directional with the three sides of the triangle ABC measured cyclically in the same direction.

122. The linear loci of Art. 120, determinable as there explained for all given values of the constant $\Sigma(a.PA)$, supply obvious solutions of the four following very general problems analogous to those of Art. 119—

Given the positions and sides, positive and negative, of any system of lines A, B, C, D, &c., and the magnitudes and signs, positive or negative, of any corresponding system of multiples a, b, c, d, &c., to determine—

1°. *On a given line the point P for which the sum $\Sigma(a.PA)$ shall have any given value, positive, negative, evanescent, or infinite.*

2°. *On a given circle the point P for which the sum $\Sigma(a.PA)$*

shall have the maximum, the minimum, or any intermediate given value.

In the particular case, when, as explained in the closing paragraph of that article (120), the particulars of the lines and multiples are such that the sum $\Sigma(a.PA)$ has the same value for every position of P not at infinity, then such problems are of course indeterminate for that particular value, and impossible at a finite distance for every other value of the sum.

123. Since in the particular case when the several segments A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., determined as in (120), on the several lines A, B, C, D, &c. are parallel, equal, and co-directional with the several sides of a polygon measured cyclically in the same direction, then, by (117), the sum $\Sigma(a.PA)$ has the same constant value for every position of P not at infinity, which value $= 0$ when the lines pass through a common point. Hence—

When a number of fixed lines A, B, C, D, &c. are parallel to the several sides a, b, c, d, &c. of any polygon, and that their positive and negative sides correspond to those of the sides of the polygon measured cyclically in either common direction, then for every point P not at infinity the sum $\Sigma(a.PA)$ is constant, and $= 0$ when the lines pass through a common point (see 113, 4°.).

When the polygon is equilateral, since then $a = b = c = d$, &c., therefore $\Sigma(a.PA) = a.\Sigma(PA)$, and therefore the sum $\Sigma(PA)$ is constant for every point not at infinity. Hence—

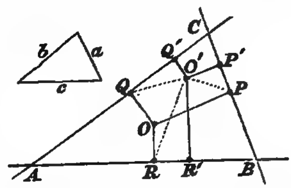
When a number of fixed lines A, B, C, D, &c. are parallel to the several sides of any equilateral polygon, and that their positive and negative sides correspond to those of the sides of the polygon measured cyclically in either common direction, then for every point P not at infinity the sum $\Sigma(PA)$ is constant, and $= 0$ when the lines pass through a common point.

Of all equilateral polygons of any order, one, the regular, being also equiangular, the term "equilateral" may therefore be replaced by "equiangular" in the statement of the latter property, the altered however being but a particular case of the original property, and no new principle of any kind being involved or expressed in the change.

124. The general property of the preceding article supplies ready solutions of the two following problems—

Given three points, or two points and a line, P, Q, R, to determine the point O for which the sum $a.OP+b.OQ+c.OR$ shall be the minimum; a, b, c being any three positive multiples no one of which is greater than the sum or less than the difference of the other two.

For, if O be the point for which the three perpendiculars at P, Q, R to OP, OQ, OR in the former case, or the two perpendiculars at P and Q to OP and OQ with the line R in the latter case, determine a triangle ABC similar to that determined by the three multiples a, b, c, and including O within its area; then if O' be any other point, and $O'P'$, $O'Q'$, $O'R'$ the three perpendiculars from it upon the three sides of ABC, since, by the preceding,



$$a.OP+b.OQ+c.OR=a.O'P'+b.O'Q'+c.O'R',$$

therefore $a.OA+b.OB+c.OC<a.O'P+b.O'Q+c.O'R$ in the former case, and $<a.O'P+b.O'Q+c.O'R'$ in the latter case, and therefore O in either case is the point required; but O is the common intersection of the three known circles QOR, ROP, POQ in the former case, and the intersection of the two known directions PO and QO in the latter case, and therefore &c.

When the three given multiples a, b, c are incapable of forming a triangle, the above method of determining O of course fails in both cases, but it is easily seen, at once without any construction, that if any of the three multiples a, b, c in the former case, or either of the two a and b corresponding to the two points P and Q in the latter case be $=$ or $>$ the sum of the other two, the point itself corresponding to that multiple is that for which the sum $a.OP+b.OQ+c.OR$ is the minimum.

But if the multiple c corresponding to the line R in the latter case be $=$ or $>$ the sum of the other two a and b corresponding to the two points P and Q, then through the required point O as before is easily seen without any construction to be on the line R, to determine its position on that line, that is, the

position of the point O on R for which the sum $a.OP+b.OQ$ is the minimum, is a problem incapable of solution by the geometry of the point, line, and circle.

125. In the linear loci of Art. 120 the several distances PA, PB, PC, PD, &c. need not be measured perpendicularly to the several lines A, B, C, D, &c.; they might be measured in directions inclined to them at any constant angles α, β, γ, δ, &c., and the several conclusions there established, with some slight and obvious modifications, would be true for the oblique as well as for the perpendicular distances.

For, PA, PB, PC, PD, &c. being the several oblique distances, and a, b, c, d, &c. as before the several corresponding multiples, if PA_1, PB_1, PC_1, PD_1, &c. be the several perpendicular distances, and a_1, b_1, c_1, d_1, &c. a system of multiples corresponding to them, having to the original multiples a, b, c, d, &c. the constant ratios of the several oblique to the corresponding perpendicular distances; then, since for every position of P not at infinity $a.PA=a_1.PA_1$, $b.PB=b_1.PB_1$, $c.PC=c_1.PC_1$, $d.PD=d_1.PD_1$, &c., therefore $\Sigma(a.PA)=\Sigma(a_1.PA_1)$, and therefore &c., the multiples for the perpendiculars being simply those for the oblique distances divided by the sines of the constant angles of inclination.

By virtue of the above the four general problems of Art. 122 may be still further generalised, by the substitution for perpendicular of oblique distances measured in any given directions from the required point P to the given lines A, B, C, D, &c.

126. If the several oblique distances PA, PB, PC, PD, &c. in the preceding, were all measured in the same absolute direction, their several points of meeting A, B, C, D, &c. with the several fixed lines would then lie in a line L passing through P parallel to the direction, and the sum $\Sigma(a.PA)$ would (Art. 80) $=\Sigma(a).PO$, when O is the mean centre on the line L of the system of points A, B, C, D, &c. for the system of multiples a, b, c, d, &c. Hence, by the preceding—

For a variable line L moving parallel to itself in any constant direction, and intersecting the several fixed lines of a polygram of any form in a system of variable points A, B, C, D, &c.

1°. *The locus of the mean centre O of the system of points A, B, C, D, &c. for any system of multiples a, b, c, d, &c. is a line M whose position depends on the direction of L.*

2°. *The locus, more generally, of the point P on L for which the sum $\Sigma\,(a.PA)$ has any constant value, positive or negative, is a line N parallel to M and distant from it in the direction of L by the interval $NM = \Sigma\,(a.PA) \div \Sigma\,(a)$.*

In the particular case when $a=b=c=d$, &c., the several lines M, loci of O for different directions of L, are termed, from the analogy of the circle, *diameters* of the polygram. The latter being given, the position of the particular diameter M corresponding to any given direction of L is determined by drawing any two lines L_1 and L_2 parallel to the given direction, taking the two mean centres O_1 and O_2 of the two systems of points A_1, B_1, C_1, D_1, &c. and A_2, B_2, C_2, D_2, &c., in which they intersect the several lines of the figure, and drawing the indefinite line O_1O_2. In the particular case where all the lines of the figure pass through a common point O, as every diameter M, corresponding to every direction of L, passes evidently through it, a single other point O_1 is therefore sufficient to determine each particular diameter in that case. Remarks precisely similar apply, of course, when the several multiples a, b, c, d, &c. have any values whatever.

In an equilateral triangle the several diameters of the figure envelope the inscribed circle; this very particular case of a much more general property, to be given in a future Chapter, is left for the present as an exercise to the reader.

127. The general property of the preceding article, combined with that of Art. 80, Cor. 1°., may be employed for the solution of the following very general problem—

Given any system of lines A, B, C, D, &c. and any corresponding system of multiples a, b, c, d, &c. to determine the point P for which the sum $\Sigma\,(a.PA^2)$ is the minimum.

Drawing arbitrarily any two parallel lines L_1 and L_2 intersecting the entire system of lines A, B, C, D, &c. at the system of angles α, β, γ, δ, &c.; taking the two mean centres O_1 and O_2 of the two systems of intersections A_1, B_1, C_1, D_1, &c. and A_2, B_2, C_2, D_2, &c. for the system of multiples $a \div \sin^2\alpha$,

$b \div \sin^2\beta$, $c \div \sin^2\gamma$, $d \div \sin^2\delta$, &c.; drawing then the indefinite line O_1O_2 intersecting the entire system of lines A, B, C, D, &c. at the system of angles α', β', γ', δ', &c.; and taking finally the mean centre O' of the system of intersections A', B', C', D', &c. for the system of multiples $a \div \sin^2\alpha'$, $b \div \sin^2\beta'$, $c \div \sin^2\gamma'$, $d \div \sin^2\delta'$, &c.; the point O' thus determined is that required.

For, by (80, Cor. 1°.), O' is the point on the line O_1O_2 for which the sum $\Sigma\,(a.PA^2)$ is the minimum for points confined to that line; and supposing, if possible, a point I not on that line were that for which it were absolutely the minimum, the line L passing through I parallel to L_1 and L_2 would intersect the line O_1O_2 at a point O, which, by the preceding, would be the mean centre for the system of multiples $a \div \sin^2\alpha$, $b \div \sin^2\beta$, $c \div \sin^2\gamma$, $d \div \sin^2\delta$, &c. of the system of points in which it would intersect the system of lines A, B, C, D, &c., and for which therefore, by (80, Cor. 1°.), the sum $\Sigma\,(a.PA^2)$ would be the minimum for points confined to the line L, and consequently less than for the point I, which therefore could not, as supposed, be off the line O_1O_2; and therefore &c.

It is easy to see from the more general property (98, Cor. 1°.), that the point P, however determined, for which the sum $\Sigma\,(a.PA^2)$ is the minimum, is the mean centre of the feet of the several perpendiculars PA, PB, PC, PD, &c. for the system of multiples a, b, c, d, &c.

128. We shall conclude the present Chapter with a direct demonstration of the general property of Art. 120, not based like that there given upon any property of polygons, but resulting immediately from the nature of independent lines; the following general theorem, analogous to that established in Art. 85 for any system of points, may be regarded as the basis of the direct demonstration—

If A, B, C, D, &c. be any system of lines, disposed in any manner, but none infinitely distant, and a, b, c, d, &c. any system of corresponding multiples, positive or negative, but none infinitely great, then for every three points P, Q, R in a line, supposed all at a finite distance,

$$QR.\Sigma\,(a.PA) + RP.\Sigma\,(a.QA) + PQ.\Sigma\,(a.RA) = 0,$$

the distances under the symbols of summation being measured in

directions inclined at any constant angles α, β, γ, δ, *&c. to the several lines* A, B, C, D, *&c.*

For, the three points P, Q, R being by hypothesis in a line, therefore, for the several lines A, B, C, D, &c., by (82, Cor. 4°.),

$$QR.PA + RP.QA + PQ.RA = 0,$$
$$QR.PB + RP.QB + PQ.RB = 0,$$
$$QR.PC + RP.QC + PQ.RC = 0,$$
$$QR.PD + RP.QD + PQ.RD = 0, \text{ \&c.},$$

from which, multiplying horizontally by a, b, c, d, &c. and then adding vertically, the above relation at once results, and from it the following consequences may be immediately inferred—

1°. *When two of the three sums* $\Sigma(a.QA)$ *and* $\Sigma(a.RA) = 0$, *the third* $\Sigma(a.PA)$ *also* $= 0$; this is evident, as the three coefficients QR, RP, PQ are by hypothesis all finite. *Hence the locus of a variable point* P *for which the sum* $\Sigma(a.PA) = 0$ *is a line*, the central axis I of the system for the particulars of the case.

2°. *When two of the three sums* $\Sigma(a.QA)$ *and* $\Sigma(a.RA)$ *have equal values, the third* $\Sigma(a.PA)$ *has the same value;* this is evident, as the sum of the three coefficients QR, RP, PQ is always $= 0$ (Art. 78). *Hence the locus of a variable point* P *for which the sum* $\Sigma(a.PA)$ *has any constant value, positive or negative, is a line* L *parallel to the central line* I; for if it met the latter at any finite distance, the sum $\Sigma(a.PA)$ for the point of intersection would have at once the two different values corresponding to the two lines.

3°. *When one of the three sums* $\Sigma(a.RA) = 0$, *then for the other two* $\Sigma(a.PA) : \Sigma(a.QA) = PR : QR = PI : QI$; this is evident, as R, by 1°., is then on the central axis I. *Hence for every point* P *on any line* L *parallel to* I, *the constant sum* $\Sigma(a.PA)$ *is proportional in magnitude and sign to the distance* LI; these are the principal results for the general case as otherwise established in Art. 120.

4°. *In the particular case when the central axis* I *of the system is at infinity, the sum* $\Sigma(a.PA)$ *has the same value for every position of* P *at a finite distance;* for since, by 3°., for every two points P and Q at a finite distance $\Sigma(a.PA) : \Sigma(a.QA) = PI : QI$, whatever be the position of I, therefore for every two points

P and Q at a finite distance $\Sigma(a.PA) : \Sigma(a.QA) = 1$ when I is at infinity, and therefore &c.

5°. *When the sum* $\Sigma(a.PA)$ *has the same value, finite or evanescent, for three points* P, Q, R *not in the same line, it has the same value for every point* O *at a finite distance;* for having the same value for the three points P, Q, R, it has it, by 2°., for the three points X, Y, Z, in which the three lines OP, OQ, OR intersect with the three QR, RP, PQ, and having it for each pair of points P and X, Q and Y, R and Z, it has it, by the same, for the point O which is in the same line with each pair; and therefore &c.

6°. *The particulars of the system being all given, the position of the central axis* I *may be determined rapidly as follows:* assuming arbitrarily any three points P, Q, R not in the same line, and dividing the three distances QR, RP, PQ at X, Y, Z respectively, so that in magnitude and sign

$$QX : RX = \Sigma(a.QA) : \Sigma(a.RA),$$
$$RY : PY = \Sigma(a.RA) : \Sigma(a.PA),$$
$$PZ : QZ = \Sigma(a.PA) : \Sigma(a.QA);$$

the three points X, Y, Z thus determined lie, by 3°., on the central axis I of the system, and therefore &c.; when the three sums $\Sigma(a.PA)$, $\Sigma(a.QA)$, $\Sigma(a.RA)$ have the same value, the three points X, Y, Z being then at infinity or indeterminate, according as the common value is finite or evanescent, so also is the central axis.

CHAPTER VIII.

ON COLLINEAR AND CONCURRENT SYSTEMS OF POINTS AND LINES.

129. SYSTEMS of points ranged on lines, and of lines passing through points, enter largely into the investigations of modern geometry, and are distinguished by appropriate names, as follows:

A system of points ranged along a line is termed a *collinear* system, the figure they constitute a *row* of points, and the line on which they lie the *base* or *axis* of the row. A system of lines passing through a point is termed a *concurrent* system, the figure they constitute a *pencil* of lines, or *rays* as they are sometimes called, and the point through which they pass the *vertex* or *centre* or *focus* of the pencil. The terms "Ray," "Pencil," and "Focus," have been introduced into geometry from the science of Optics.

The axis of a row of points, or the centre of a pencil of lines, might be at infinity; in the former case the points of the row would, of course, be all at infinity, and in the latter case the lines of the pencil would (16) be all parallel; but in no other respects is there any difference between these particular and the general cases, when the axis of the row is any line whatever, and the centre of the pencil any point whatever.

Two points of a row or rays of a pencil determine, of course, the axis or vertex of the row or pencil to which they belong.

130. Two rows of any common number of points on different axes, or pencils of any common number of rays through different centres, A, B, C, D, &c., and A', B', C', D', &c., whose constituents correspond in pairs A to A', B to B', C to C', D to D', &c., are said to be *in perspective*, in the former case when the several lines of connexion AA', BB' CC' DD', &c., of pairs of

corresponding points are concurrent, and in the latter case when the several points of intersection AA', BB', CC', DD', &c., of pairs of corresponding lines are collinear. In the former case the centre of the pencil determined by the several concurrent connectors is termed *the centre of perspective* of the rows, and in the latter case the axis of the row determined by the several collinear intersections is termed *the axis of perspective* of the pencils. Every two rows of points determined on different axes by the same pencil of rays, and every two pencils of rays determined at different centres by the same row of points, are evidently in perspective; the centre of the determining pencil being the centre of perspective of the rows in the former case, and the axis of the determining row being the axis of perspective of the pencils in the latter case.

The centre of perspective of two rows in perspective, or the axis of perspective of two pencils in perspective, might be at infinity; in the former case the several connectors AA', BB', CC', DD', &c. being all parallel, the two rows of points would be similar (32), and in the latter case the several pairs of corresponding rays A and A', B and B', C and C', D and D', &c., being two and two parallel, the two pencils would be similar, and at once similarly and oppositely placed (33). In these particular cases of perspective the two rows or pencils $ABCD$ &c., and $A'B'C'D'$ &c., are said also to be *projections* of each other; though both terms "perspective" and "projection" are often applied indifferently as well to the general as to the particular case, and, as we shall see in the sequel, to other figures as well as to rows and pencils.

131. Every two rows or pencils of but *two* points or rays each having different axes or vertices being, of course, necessarily in perspective, however circumstanced as to position, absolute or relative, or whichever way regarded as corresponding two and two. Hence for two segments or angles AB and $A'B'$ having different axes or vertices, the two points of intersection, or lines of connection, of AB with $A'B'$, and, of AB' with $A'B$, are termed respectively *the two centres of perspective of the segments*, or, *the two axes of perspective of the angles*—names, at once convenient and expressive, by which to designate a pair

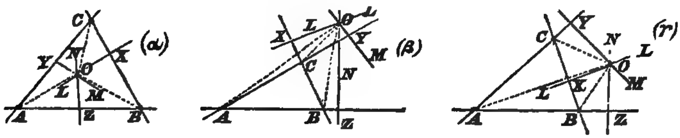
of points or lines of very frequent occurrence in geometrical research.

Reserving for future chapters the remarkable developments of modern Geometry as regards collinear and concurrent systems in general, we shall devote the present chapter to the consideration of some of their most important properties as regards the sides and angles of rectilinear figures in general, and of triangles in particular.

132. *When three lines LX, MY, NZ intersecting at right angles the three sides BC, CA, AB of any triangle ABC are concurrent, they divide them at the three parts of meeting X, Y, Z so as to fulfil the relation*

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2) = 0$$

and, conversely, when they divide them at the three points of meeting so as to fulfil the above relation, they are concurrent.



To prove the first or direct part; if O be the point of concurrence of the three lines LX, MY, NZ, then joining OA, OB, OC, since (Euclid I, 47, Cor.) $(BX^2 - CX^2) = (BO^2 - CO^2)$, $(CY^2 - AY^2) = (CO^2 - AO^2)$, $(AZ^2 - BZ^2) = (AO^2 - BO^2)$, therefore

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2)$$
$$= (BO^2 - CO^2) + (CO^2 - AO^2) + (AO^2 - BO^2) = 0$$

as above stated. And to prove the second or converse part; if O be the point of intersection of any two of them LX and MY, and Z' the point at which the parallel through O to the third NZ intersects the line AB to which the third is perpendicular; then since by the first part

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ'^2 - BZ'^2) = 0,$$

and since by hypothesis

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2) = 0;$$

therefore $(AZ'^2 - BZ'^2) = (AZ^2 - BZ^2)$, and therefore $Z' = Z$, which, of course, could not be the case unless, as above stated, NZ passed through O.

A relation of exactly the same form, and proved in precisely the same manner as the above, connects the several pairs of segments into which the several sides of any polygon are divided by every concurrent system of perpendiculars to them. But the converse property which establishes the relation as *a criterion of concurrence* of the several perpendiculars *is true only for the triangle.* The reasoning by which it was inferred as above for that case from the direct property evidently proving only for the general case of any order (n), that when $(n-1)$ of them pass through a common point the n^{th} passes through the same point.

The relation itself may evidently be written also in the following form, viz.—

$$BX^2 + CY^2 + AZ^2 = CX^2 + AY^2 + BZ^2,$$

which in cases of numerical calculation is sometimes more convenient than the original.

133. The following are a few examples of the application of the preceding relation as a criterion of the concurrence of three lines perpendiculars at three points X, Y, Z to the three sides of a triangle ABC.

Ex. 1°. *The three perpendiculars at the middle points of the sides of a triangle are concurrent.*

For, since here by hypothesis, $BX = CX$, $CY = AY$, $AZ = BZ$, therefore the criterion relation $(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2) = 0$ is satisfied identically in the simplest manner of which it is susceptible, and therefore &c.

Ex. 2°. *The three perpendiculars through the vertices to the opposite sides of a triangle are concurrent.*

For, since here, Euc. I. 47,

$$(BX^2 - CX^2) = (BA^2 - CA^2),\ (CY^2 - AY^2) = (CB^2 - AB^2),$$
$$(AZ^2 - BZ^2) = (AC^2 - BC^2);$$

therefore the criterion relation again is satisfied, and therefore &c.

Ex. 3°. *The three perpendiculars to the sides of a triangle at the internal points of contact of the three exscribed circles are concurrent.*

For, if a, b, c be the three sides and s the semi-perimeter of the triangle, then since, Euc. IV. Appendix,

$$BZ = CY = (s - a),\quad CX = AZ = (s - b),\quad AY = BX = (s - c);$$

therefore, as in Ex. 1°., the criterion relation is identically satisfied, and therefore &c.

Ex. 4°. *Every two perpendiculars to sides of a triangle at points of contact of exscribed circles external to the same vertex are concurrent with the perpendicular to the opposite side at the point of contact of the inscribed circle.*

For, if A be the vertex to which the two contacts are external; then since, Euc. IV., Appendix,

$$AY = BX = (s - b),\quad AZ = CX = (s - c),\quad BZ = CY = s;$$

therefore, here again, as in the preceding example, the criterion relation is identically satisfied, and therefore &c.

Ex. 5°. *When three circles touch two and two, the three tangents at the three points of contact are concurrent.*

For, if A, B, C be the centres of the three circles, a, b, c the three radii, and X, Y, Z the three opposite points of contact, then since

$$AY = AZ = a,\quad BZ = BX = b,\quad CX = CY = c;$$

therefore, as in each of the preceding examples, the criterion relation is again identically satisfied, and therefore &c.

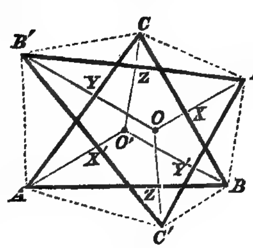
Ex. 6°. *When three perpendiculars to the sides of a triangle are concurrent, the other three equidistant from the middle points of the sides are also concurrent.*

For, if LX, MY, NZ and $L'X'$, $M'Y'$, $N'Z'$ be the two sets of perpendiculars, then since by hypothesis $BX = CX'$ and $CX = BX'$, $CY = AY'$ and $AY = CY'$, $AZ = BZ'$ and $BZ = AZ'$; therefore

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2)$$
$$= (CX'^2 - BX'^2) + (AY'^2 - CY'^2) + (BZ'^2 - AZ'^2),$$

and therefore when either equivalent = 0, so is the other; that is, when either set of perpendiculars is concurrent, so is the other.

Ex. 7°. *When the three perpendiculars from the vertices of one triangle upon the sides of another are concurrent, the three corresponding perpendiculars from the vertices of the latter upon the sides of the former are also concurrent.*



Let ABC and $A'B'C'$ be the two triangles. If $A'X$, $B'Y$, $C'Z$ pass through a common point O, then AX', BY', CZ' pass also through a common point O', and conversely. For, joining A with B' and C', or A' with B and C; B with C' and A', or B' with C and A; C with A' and B', or C' with A and B; that is, each vertex of either triangle with the two of the other it does not correspond to, then

$$\begin{aligned}
&(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2)\\
=\ &(BA'^2 - CA'^2) + (CB'^2 - AB'^2) + (AC'^2 - BC'^2)\\
=\ &(C'A^2 - B'A^2) + (A'B^2 - C'B^2) + (B'C^2 - A'C^2)\\
=\ &(C'X'^2 - B'X'^2) + (A'Y'^2 - C'Y'^2) + (B'Z'^2 - A'Z'^2),
\end{aligned}$$

but of these four equivalents the first = 0 is the condition for $A'X$, $B'Y$, $C'Z$ to pass through a common point O, and the last = 0 is the condition for AX', BY', CZ' to pass through a common point O', and therefore &c.

134. *When three points X, Y, Z lying on the three sides BC, CA, AB of any triangle ABC are collinear* (figs. α, β, γ).

a. They divide the three sides so as to fulfil the relation

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=+1.$$

b. They connect with the opposite vertices so as to fulfil the relation

$$\frac{\sin BAX}{\sin CAX}\cdot\frac{\sin CBY}{\sin ABY}\cdot\frac{\sin ACZ}{\sin BCZ}=+1;$$

and conversely, when they either divide the three sides so as to fulfil the former relation, or connect with the opposite vertices so as to fulfil the latter relation, they are collinear.

When three lines AX, BY, CZ passing through the three vertices A, B, C of any triangle ABC are concurrent (figs. α', β', γ').

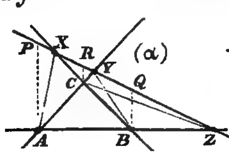
a′. They divide the three angles so as to fulfil the relation

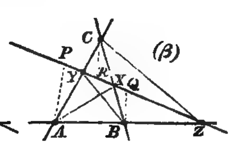
$$\frac{\sin BAX}{\sin CAX}\cdot\frac{\sin CBY}{\sin ABY}\cdot\frac{\sin ACZ}{\sin BCZ}=-1.$$

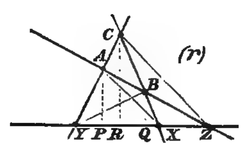
b′. They intersect the opposite sides so as to fulfil the relation

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=-1;$$

and conversely, when they either divide the three angles so as to fulfil the former relation, or intersect the opposite sides so as to fulfil the latter relation, they are concurrent.







To prove the first or direct part of a.—From the three vertices of the triangle ABC drawing three perpendiculars, or parallels in any arbitrary direction, AP, BQ, CR to meet the line containing, by hypothesis, the three points X, Y, Z, then since (Euc. VI. 4) $BX : CX = BQ : CR$, $CY : AY = CR : AP$, $AZ : BZ = AP : BQ$; therefore, as above stated, the compound

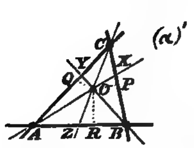
of the three antecedents $= + 1$; the reason of the positive sign appearing also from the obvious consideration that three collinear points on the sides of a triangle necessarily divide an *even* number of them *internally* (75). And to prove the second or converse part of the same.—If Z' be the point at which the line containing any two of the points X and Y meets the side AB of the triangle containing the third Z; since then, by the first part, X, Y, and Z' being collinear,

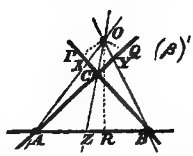
$$(BX : CX).(CY : AY).(AZ' : BZ') = + 1,$$

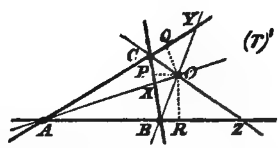
and since also, by hypothesis,

$$(BX : CX).(CY : AY).(AZ : BZ) = + 1,$$

therefore $AZ' : BZ' = AZ : BZ$ in magnitude and sign, and therefore (75) $Z' = Z$, so that, as above stated, Z is collinear with X and Y.







To prove the first or direct part of a'.—From the point O through which, by hypothesis, the three lines AX, BY, CZ concur, letting fall three perpendiculars, or isoclinals at any arbitrary inclination, OP, OQ, OR upon the three sides BC, CA, AB of the triangle ABC; then since (61)

$$\sin BAX : \sin CAX = - OR : OQ, \ \sin CBY : \sin ABY = - OP : OR,$$
$$\sin ACZ : \sin BCZ = - OQ : OP,$$

therefore, as above stated, the compound of the three antecedents $= - 1$; the reason of the negative sign appearing also from the obvious consideration that three concurrent lines through the vertices of a triangle necessarily divide an *odd* number of the angles *internally* (75). And to prove the second or converse part of the same.—If CZ' be the line by which the point O, common to any two of the lines AX and BY, connects with the vertex C of the triangle through which the third CZ passes; since then, by the first part, AX, BY, and CZ' being concurrent,

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ' : \sin BCZ') = - 1,$$

and since also, by hypothesis,

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ) = -1,$$

therefore $\sin ACZ' : \sin BCZ' = \sin ACZ : \sin BCZ$ in magnitude and sign, and therefore (75) $CZ' = CZ$, so that, as above stated, CZ is concurrent with AX and BY.

Relations of exactly the same form, and proved in precisely the same manner as the above (a and a'), connect the several pairs of segments into which the several sides of any polygon are divided by every collinear system of points lying upon them, and into which the several angles of any polygon are divided by every concurrent system of lines passing through them; the only modification being that while, in the former case, the sign of the compound is, as above, always positive, in the latter case it is negative only when, as above, the order of the polygon is odd, but positive when it is even. The converse properties however, which establish the relations a and a' as *criteria of collinearity and concurrence* of the several points and lines *are true only for the triangle;* the reasoning by which they have been inferred, as above, for that case from the direct properties proving only for the general case of any order (n), that when $(n-1)$ of the points in the former case are collinear the n^{th} is collinear with them, and that when $(n-1)$ of the lines in the latter case are concurrent the n^{th} is concurrent with them.

To prove b and b'.—Since, by (65), whatever be the positions of X, Y, Z in the former case, or the directions of AX, BY, CZ in the latter case,

$$\frac{BX}{CX} = \frac{BA}{CA} \cdot \frac{\sin BAX}{\sin CAX}, \quad \frac{CY}{AY} = \frac{CB}{AB} \cdot \frac{\sin CBY}{\sin ABY}, \quad \frac{AZ}{BZ} = \frac{AC}{BC} \cdot \frac{\sin ACZ}{\sin BCZ},$$

therefore, the two compounds

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} \text{ and } \frac{\sin BAX}{\sin CAX} \cdot \frac{\sin CBY}{\sin ABY} \cdot \frac{\sin ACZ}{\sin BCZ}$$

are always equal in magnitude and similar in sign; whenever, therefore, either $= \pm 1$, so is also the other, and therefore &c.

Relations of exactly the same form with these last (b and b'), and very easily proved directly, as they too may be, connect the several pairs of segments into which, for any polygon of an *odd* order (105), the several angles are divided by their con-

nectors with collinear systems of points on the opposite sides, and the several sides at their intersections with concurrent systems of lines through the opposite vertices; but, as in the cases of a and a', the converse properties which establish the relations as *criteria of collinearity and concurrence* of the several points and lines, are, for the same reason as in their cases, true only for the triangle.

The criteria (a and b') for three points X, Y, Z on the sides of a triangle to be collinear and to connect with the opposite vertices by three concurrent lines AX, BY, CZ, and the criteria (b and a') for three lines AX, BY, CZ through the vertices of a triangle to be concurrent and to intersect with the opposite sides at three collinear points X, Y, Z, being in both cases identical, if signs be disregarded or unknown; should any ambiguity arise in consequence, as to which of the two relations in either case is indicated by the fulfilment of the criterion in any particular instance, in which the signs of the compound ratios are not explicitly given or known; the obvious consideration, on which the difference of sign in each case depends, that an *odd* number of the points or lines must be *external* to their respective sides or angles for *collinearity*, and *internal* to them for *concurrence*, is sufficient always to remove it.

135. The following is an obvious corollary from, or rather indeed a different manner of, stating the two general properties a' and b of the preceding article, viz.,

When three points P, Q, R, however situated, connect with the three vertices A, B, C of a triangle ABC by three lines AP, BQ, CR which are either concurrent or collinearly intersectant with the opposite sides, the three pairs of perpendiculars PP' and PP'', QQ' and QQ'', RR' and RR'' from them upon the three pairs of sides containing the respective vertices are connected by the relation

$$\frac{PP'}{PP''}\cdot\frac{QQ'}{QQ''}\cdot\frac{RR'}{RR''}=\pm 1,$$

and conversely, when the three pairs of perpendiculars from them upon the sides of the three angles of the triangle are connected by the above relation, the three lines connecting them with the

corresponding vertices are either concurrent or collinearly intersectant with the opposite sides.

For, whatever be the positions of P, Q, R, since (61)

$$PP' : PP'' = -\sin PAB : \sin PAC,$$
$$QQ' : QQ'' = -\sin QBC : \sin QBA,$$
$$RR' : RR'' = -\sin RCA : \sin RCB,$$

therefore the two compounds

$$\frac{PP'}{PP''}\cdot\frac{QQ'}{QQ''}\cdot\frac{RR'}{RR''} \text{ and } \frac{\sin BAP}{\sin CAP}\cdot\frac{\sin CBQ}{\sin ABQ}\cdot\frac{\sin ACR}{\sin BCR}$$

are always equal in magnitude and opposite in sign, and therefore when either $= \pm 1$ the other then $= \mp 1$, and therefore &c.

136. Two very important conclusions, one respecting points at infinity, the other respecting parallel lines, result immediately from the general relations a or b, and a' or b' of Art. 134, regarded as criteria of the collinearity of three points X, Y, Z on the sides, and of the concurrence of these lines AX, BY, CZ through the vertices, of a triangle ABC—

1°. Every three points X, Y, Z at infinity evidently divide the three sides BC, CA, AB of every triangle ABC whose directions pass through them, so as to fulfil identically the relation

$$(BX : CX).(CY : AY).(AZ : BZ) = +1,$$

and as evidently connect with the opposite vertices, so as to fulfil identically the relation

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ) = +1,$$

therefore, by relation a or b, they are collinear, and therefore—

Every three, and therefore all, points at infinity are collinear.

2°. Every three parallel lines AX, BY, CZ evidently divide the three angles BAC, CBA, ACB of every triangle ABC whose vertices lie on them, so as to fulfil identically the relation

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ) = -1,$$

and as evidently intersect with the opposite sides, so as to fulfil the relation

$$(BX : CX).(CY : AY).(AZ : BZ) = -1,$$

therefore, by relation a' or b', they are concurrent, and therefore—

Every three, and therefore all, parallel lines are concurrent.

Paradoxical as these conclusions always appear when first stated, all doubt of their legitimacy has been long set at rest by the number and variety of the considerations tending to verify and confirm them.

137. In the following examples of the application of the preceding relations, as criteria of the collinearity of three points X, Y, Z on three lines, and of the concurrence of three lines AX, BY, CZ through three points, one only of the two relations equally establishing the circumstance being proved in each case, the verification *à priori* of the other may be taken as an exercise by the reader.

Ex. 1°. *Every three points of bisection of different sides of a triangle are collinear, or connect concurrently with the opposite vertices, according as an odd number of them is external or internal.*

For, since by hypothesis, $BX : CX = \pm 1$, $CY : AY = \pm 1$, $AZ : BZ = \pm 1$, according as each section is external or internal, therefore the criterion relation (a or b') for collinearity or concurrence, viz.

$$(BX : CX).(CY : AY).(AZ : BZ) = \pm 1,$$

according as an odd number of them is external or internal, is satisfied in the simplest manner of which it is susceptible, and therefore &c.

Ex. 2°. *Every three lines of bisection of different angles of a triangle are concurrent, or intersect collinearly with the opposite sides, according as an odd number of them is internal or external.*

For, since by hypothesis,

$$\sin BAX : \sin CAX = \pm 1,\ \sin CBY : \sin ABY = \pm 1,\ \sin ACZ : \sin BCZ = \pm 1,$$

according as each section is external or internal, therefore the criterion relation (a' or b) for concurrence or collinearity, viz.

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ) = \mp 1,$$

according as an odd number of them is internal or external, is satisfied in the simplest manner of which it is susceptible, and therefore &c.

Ex. 3°. *In every triangle circumscribed to a circle the three points of contact of the sides connect concurrently with the opposite vertices.*

For, if X, Y, Z be the three points of contact, then, since, by pairs of equal tangents from ABC to the circle, $AY = AZ$, $BZ = BX$, $CX = CY$, therefore, as in Ex. 1°., the criterion relation (b') for the concurrence of AX, BY, CZ is identically satisfied; it being evident, from the nature of the case, that the three points X, Y, Z must, according to circumstances, be either all internal or one internal and two external to their respective sides, and therefore &c.

Ex. 4°. *In every triangle inscribed in a circle the three tangents at the vertices intersect collinearly with the opposite sides.*

For if AX, BY, CZ be the three tangents, then, since, by pairs of equal angles between BC, CA, AB and the circle, $\sin CBY = \sin BCZ$, $\sin ACZ = \sin CAX$, $\sin BAX = \sin ABY$, therefore, as in example 2°., the criterion relation (b) for the collinearity of XYZ is identically satisfied; it being evident, from the nature of the case, that the three lines AX, BY, CZ must, under all circumstances, be external to their respective angles, and therefore &c.

Ex. 5°. *In every triangle the three perpendiculars through the vertices to the opposite sides are concurrent* (See Ex. 2°, 133).

For, if AX, BY, CZ be the three perpendiculars, then, since, by pairs of similar right-angled triangles about A, B, C as common vertices,

$$\sin ABY = \sin ACZ, \quad \sin BCZ = \sin BAX, \quad \sin CAX = \sin CBY,$$

therefore the criterion relation (a') for the concurrence of AX, BY, CZ is identically satisfied; it being evident, from the nature of the case, that, according as the triangle is acute or obtuse angled, they are either all internal or one internal and two external to their respective angles, and therefore &c.

Ex. 6°. *In every triangle the three perpendiculars through any point to the three lines connecting them with the vertices intersect collinearly with the opposite sides.*

For, if O be the point, and OX, OY, OZ the three perpendiculars through it to OA, OB, OC respectively, then, since, by (65),

$$BX : CX = (BO : CO) . (\sin BOX : \sin COX),$$

$$CY : AY = (CO : AO) . (\sin COY : \sin AOY),$$

$$AZ : BZ = (AO : BO) . (\sin AOZ : \sin BOZ);$$

and since, by pairs of perpendiculars,

$$\sin COY = \sin BOZ, \quad \sin AOZ = \sin COX, \quad \sin BOX = \sin AOY,$$

therefore the criterion relation (a) for the collinearity of XYZ is satisfied; it being evident, from the nature of the case, that they must be, according to circumstances, either all external or one external and two internal to their respective sides, and therefore &c.

Ex. 7°. *If the three sides of a triangle be reflected with respect to any line* (50), *the three lines through the vertices parallel to the reflexions of the opposite sides are concurrent.*

For, if AX, BY, CZ be the three parallels, then, since, by differences of pairs of equal angles (50),

$$\sin ABY = \sin ACZ, \quad \sin BCZ = \sin BAX, \quad \sin CAX = \sin CBY,$$

therefore the criterion relation (a') for the concurrence of AX, BY, CZ is identically satisfied; it being evident, from the nature of the case, that, according as the axis of reflexion is or is not parallel to a bisector of an

angle of the triangle, either two of them coincide with the sides of that angle or two are external and one internal to their respective angles, and therefore &c.

Ex. 8°. *If the three vertices of a triangle be reflected with respect to any line* (50), *the three lines connecting the reflexions with any point on the line intersect collinearly with the opposite sides.*

For, if A', B', C' be the three reflexions, O the point on the line, and X, Y, Z the three intersections of OA', OB', OC', with BC, CA, AB respectively, then, since, by (65),

$$BX : CX = (BO : CO) . (\sin BOX : \sin COX),$$
$$CY : AY = (CO : AO) . (\sin COY : \sin AOY),$$
$$AZ : BZ = (AO : BO) . (\sin AOZ : \sin BOZ);$$

and since, by differences of pairs of equal angles (50),

$$\sin COY = \sin BOZ, \quad \sin AOZ = \sin COX, \quad \sin BOX = \sin AOY,$$

therefore the criterion relation a, for the collinearity of X, Y, Z, is satisfied exactly as in Ex. 6°; it being evident, from the nature of the case, that, here as well as there, they must, according to circumstances, be either all external or one external and two internal to their respective sides, and therefore &c.

Ex. 9°. *When three of the six intersections of a circle with the three sides of a triangle connect concurrently with the opposite vertices, the remaining three also connect concurrently with the opposite vertices.*

For, if X, Y, Z and X', Y', Z' be the two sets of intersections, then, since, by Euc. III. 35, 36,

$$AY.AY' = AZ.AZ', \quad BZ.BZ' = BX.BX', \quad CX.CX' = CY.CY',$$

therefore

$$(AY:AZ).(BZ:BX).(CX:CY) = (AZ':AY').(BX':BZ').(CY':CX'),$$

and therefore when either equivalent $= -1$ so is the other; that is, when either set of connectors AX, BY, CZ, or AX', BY', CZ' is concurrent so is the other. As no three points on a circle could be collinear, neither equivalent could $= +1$ in this case.

Ex. 10°. *When three of the six tangents to a circle from the three vertices of a triangle intersect collinearly with the opposite sides, the remaining three also intersect collinearly with the opposite sides.*

For, if AX, BY, CZ and AX', BY', CZ' be the two sets of tangents, and a, b, c the lengths of the three chords intercepted by the circle on the three sides of the triangle, since then, by (66, Cor. 2°),

$$\sin BAX . \sin BAX' : \sin CAX . \sin CAX' = c^2 : b^2,$$
$$\sin CBY . \sin CBY' : \sin ABY . \sin ABY' = a^2 : c^2,$$
$$\sin ACZ . \sin ACZ' : \sin BCZ . \sin BCZ' = b^2 : a^2,$$

therefore

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ)$$

$$= (\sin CAX' : \sin BAX').(\sin ABY' : \sin CBY').(\sin BCZ' : \sin BCZ'),$$

and therefore when either equivalent $= +1$ so is the other; that is, when either set of intersections X, Y, Z or X', Y', Z' is collinear so is the other. As no three tangents to a circle could be concurrent, neither equivalent could $= -1$ in this case.

Ex. 11°. *When three points on the sides of a triangle are either collinear or concurrently connectant with the opposite vertices, the other three equally distant from the bisections of the sides are also either collinear or concurrently connectant with the opposite vertices.*

For, if X, Y, Z and X', Y', Z' be the two sets of points, then, since, by hypothesis, $BX = CX'$ and $CX = BX'$, $CY = AY'$ and $AY = CY'$, $AZ = BZ'$ and $BZ = AZ'$, therefore

$$(BX : CX).(CY : AY).(AZ : BZ) = (CX' : BX').(AY' : CY').(BZ' : AZ'),$$

and therefore when either equivalent $= \pm 1$ so is also the other; that is, when either set of points X, Y, Z or X', Y', Z' is collinear, or, when either set of lines AX, BY, CZ or AX', BY', CZ' is concurrent, so is also the other, and therefore &c.

Ex. 12°. *When three lines through the vertices of a triangle are either concurrent or collinearly intersectant with the opposite sides, the other three equally inclined to the bisectors of the angles are also either concurrent or collinearly intersectant with the opposite sides.*

For, if AX, BY, CZ and AX', BY', CZ' be the two sets of lines, then since by hypothesis $BAX = CAX'$ and $CAX = BAX'$, $CBY = ABY'$ and $ABY = CBY'$, $ACZ = BCZ'$ and $BCZ = ACZ'$, therefore

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ)$$

$$= (\sin CAX' : \sin BAX').(\sin ABY' : \sin CBY').(\sin BCZ' : \sin ACZ'),$$

and therefore when either equivalent $= \mp 1$ so is also the other, that is when either set of lines AX, BY, CZ or AX', BY', CZ' is concurrent, or when either set of points X, Y, Z or X', Y', Z' is collinear, so is also the other, and therefore &c.

Ex. 13°. *When three lines through the vertices of a triangle are concurrent, the six bisectors of the three angles they determine intersect with the corresponding sides of the triangle at six points, every three of which on different sides are either collinear or concurrently connectant with the opposite vertices, according as an odd number of them is external or internal.*

For, if O be the point of concurrence of the lines, and X, Y, Z the intersections with the sides of the triangle of any three of the six bisectors of the three angles BOC, COB, AOB, then, since Euc. VI. 3,

$$BX : CX = \pm BO : CO,\quad CY : AY = \pm CO : AO,\quad AZ : BZ = \pm AO : BO,$$

according as each bisector is external or internal, therefore

$$(BX : CX) . (CY : AY) . (AZ : BZ) = \pm 1,$$

according as an odd number of them is external or internal, and therefore &c.

Ex. 14°. *When three points on the sides of a triangle are collinear, the six bisections of the three segments they determine connect with the corresponding vertices of the triangle by six lines, every three of which through different vertices are either concurrent or collinearly intersectant with the opposite sides, according as an odd number of them is internal or external.*

For, if P, Q, R be the three collinear points, and AX, BY, CZ any three of the six lines through A, B, C bisecting externally and internally the three intercepted segments QR, RP, PQ, since then, by (65, Cor 3°.),

$$\sin BAX : \sin CAX = \pm AQ : AR, \ \sin CBY : \sin ABY = \pm BR : BP,$$

$$\sin ACZ : \sin BCZ = \pm CP : CQ,$$

according as each bisector divides its angle of the triangle externally or internally, and since, by (a),

$$(BP : CP) . (CQ : AQ) : (AR : BR) = + 1,$$

the three points P, Q, R being by hypothesis collinear, therefore

$$(\sin BAX : \sin CAX) . (\sin CBY : \sin ABY) . (\sin ACZ : \sin BCZ) = \mp 1,$$

according as an odd number of the bisectors is internal or external, and therefore &c.

N.B. With respect to this last example and all others of the same kind, it is to be observed that, since, of the three segments intercepted on any line by the three angles of any triangle, *two* are always comprehended in the *internal* and *one* always in the *external* regions of the intercepted angle, (see figs. α, β, γ, Art. 134), therefore an odd number of sections of either kind for the segments corresponds always to an odd number of sections of the other kind for the angles, and conversely.

138. The two last Examples, 13° and 14°, of the preceding Article are particular cases of the two following general properties—

1°. *When three points X, Y, Z on the sides of a triangle ABC are collinear or connect concurrently with the opposite vertices, their connectors OX, OY, OZ with any arbitrary point O divide the three angles BOC, COA, AOB subtended at that point by the sides of the triangle, so as to fulfil the relation*

$$\frac{\sin BOX}{\sin COX} \cdot \frac{\sin COY}{\sin AOY} \cdot \frac{\sin AOZ}{\sin BOZ} = \pm 1,$$

and conversely, when they connect with any point O so as to fulfil the above relation they are collinear or connect concurrently with the opposite vertices.

2°. *When three lines AX, BY, CZ through the vertices of a triangle ABC are concurrent or intersect collinearly with the opposite sides, their intersections X, Y, Z with any arbitrary line L divide the three segments QR, RP, PQ intercepted on that line by the angles of the triangle, so as to fulfil the relation*

$$\frac{QX}{RX}\cdot\frac{RY}{PY}\cdot\frac{PZ}{QZ}=\pm 1,$$

and conversely, when they intersect with any line L so as to fulfil the above relation they are concurrent or intersect collinearly with the opposite sides.

For, whatever be the positions of X, Y, Z in 1°, since, by (65), disregarding signs for a moment,

$$\sin BOX : \sin COX = (BX : CX).(CO : BO),$$
$$\sin COY : \sin AOY = (CY : AY).(AO : CO),$$
$$\sin AOZ : \sin BOZ = (AZ : BZ).(BO : AO);$$

and since, evidently, the internal and external sections of BC, CA, AB and of BOC, COA, AOB always correspond, therefore the two compounds,

$$(\sin BOX : \sin COX).(\sin COY : \sin AOY).(\sin AOZ : \sin BOZ)$$

and $$(BX : CX).(CY : AY).(AZ : BZ)$$

are always equal in magnitude and similar in sign, and therefore when either of them $= \pm 1$ so is the other also, which, by relations a and b' of the preceding, proves both parts of 1°. And whatever be the positions of AX, BY, CZ in 2°, since, by the same, again disregarding signs for a moment,

$$QX : RX = (QA : RA).(\sin CAX : \sin BAX),$$
$$RY : PY = (RB : PB).(\sin ABY : \sin CBY),$$
$$PZ : QZ = (PC : QC).(\sin BCZ : \sin ACZ);$$

and since, by (a), the three points P, Q, R being collinear,

$$(QA : RA).(RB : PB).(PC : QC) = +1;$$

therefore, remembering (see note to the preceding Article) that the odd number of sections of either kind for QR, RP, PQ

corresponds always to the odd number of sections of the other kind for BAC, CBA, ACB, and conversely, the two compounds

$$(QX : RX).(RY : PY).(PZ : QZ)$$

and

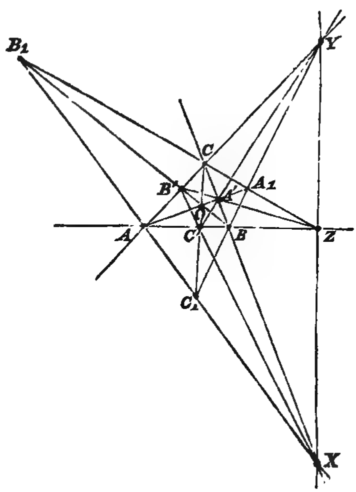
$$(\sin CAX : \sin BAX).(\sin ABY : \sin CBY).(\sin BCZ : \sin ACZ),$$

are always equal in magnitude and opposite in sign, and therefore when either of them $= \pm 1$ the other then $= \mp 1$, which, by relations a' and b of the preceding, prove both parts of 2°, and therefore &c.

139. The next example is given separately from the utility of the double property in the modern geometry of the triangle.

a. When three lines through the vertices of a triangle are concurrent, their three points of intersection with the opposite sides determine an inscribed triangle whose sides intersect collinearly with those of the original to which they correspond.

b. When three points on the sides of a triangle are collinear, their three lines of connection with the opposite vertices determine an exscribed triangle whose vertices connect concurrently with those of the original to which they correspond.



To prove *a.*—If ABC be the original triangle, $A'B'C'$ *any* inscribed triangle, and X, Y, Z the three intersections of their three pairs of corresponding sides BC and $B'C'$, CA and $C'A'$, AB and $A'B'$, then, whatever be the positions of $A'B'C'$, since, by (134, *a.*),

$$BX : CX = (BC' : AC').(AB' : CB'),$$

$$CY : AY = (CA' : BA').(BC' : AC'),$$

$$AZ : BZ = (AB' : CB').(CA' : BA'),$$

therefore, in all cases,

$$(BX : CX).(CY : AY).(AZ : BZ) = (CA' : BA')^2.(AB' : CB')^2.(BC' : AC')^2,$$

and therefore, as above stated, when AA', BB', CC' are concurrent X, Y, Z are collinear, and conversely, both equivalents being then $=+1$.

To prove *b*.—If ABC be the original triangle, $A_1B_1C_1$ *any* exscribed triangle, and X, Y, Z the three intersections of their three pairs of corresponding sides BC and B_1C_1, CA and C_1A_1, AB and A_1B_1, then, whatever be the directions of AX, BY, CZ, since, by (134, a'.),

$$\sin BAA_1 : \sin CAA_1 = -(\sin BCZ : \sin ACZ).(\sin ABY : \sin CBY),$$
$$\sin CBB_1 : \sin ABB_1 = -(\sin CAX : \sin BAX).(\sin BCZ : \sin ACZ),$$
$$\sin ACC_1 : \sin BCC_1 = -(\sin ABY : \sin CBY).(\sin CAX : \sin BAX),$$

therefore, in all cases,

$$(\sin BAA_1 : \sin CAA_1).(\sin CBB_1 : \sin ABB_1).(\sin ACC_1 : \sin BCC_1)$$
$$= -(\sin CAX : \sin BAX)^2.(\sin ABY : \sin CBY)^2.(\sin BCZ : \sin ACZ)^2,$$

and therefore, as above stated, when X, Y, Z are collinear AA_1, BB_1, CC_1 are concurrent, and conversely, both equivalents being then $=-1$.

Cor. 1°. When, as in *a*, the three lines AA', BB', CC' are concurrent, and the three points X, Y, Z therefore collinear, or conversely, it is easy to see that then always

$$\frac{BX}{CX} = -\frac{BA'}{CA'}, \quad \frac{CY}{AY} = -\frac{CB'}{AB'}, \quad \frac{AZ}{BZ} = -\frac{AC'}{BC'},$$

relations which give at once, numerically, the positions of the three points X, Y, Z when those of the three A', B', C' are known, and conversely.

For, by (134) *a* and *b'*, the common values of the three pairs of equivalents are expressed alike by the three compounds,

$$(BC' : AC').(AB' : CB'), \quad (CA' : BA').(BC' : AC'),$$
$$(AB' : CB').(CA' : BA'),$$

respectively, and therefore &c.

Cor. 2°. When, as in *b*, the three points X, Y, Z are collinear, and the three lines AA_1, BB_1, CC_1 therefore concurrent, or conversely, it is easy to see that then always

$$\frac{\sin BAX}{\sin CAX} = -\frac{\sin BAA_1}{\sin CAA_1}, \quad \frac{\sin CBY}{\sin ABY} = -\frac{\sin CBB_1}{\sin ABB_1},$$
$$\frac{\sin ACZ}{\sin BCZ} = -\frac{\sin ACC_1}{\sin BCC_1},$$

relations which give at once, numerically, the directions of the three lines AA_1, BB_1, CC_1, when those of the three AX, BY, CZ are known, and conversely.

For, by (134) b and a', the common values of the three pairs of equivalents are expressed alike by the three compounds

$$(\sin BCZ : \sin ACZ).(\sin ABY : \sin CBY),$$
$$(\sin CAX : \sin BAX).(\sin BCZ : \sin ACZ),$$
$$(\sin ABY : \sin CBY).(\sin CAX : \sin BAX),$$

respectively, and therefore &c.

COR. 3°. From the preceding relations it may be easily shown, that, for the same triangle ABC, the same line XYZ corresponds always to the same point O, and the same point O to the same line XYZ, in the two properties a and b.

For, if XYZ be given, then since, by the relations of Cor. 1°, the three sets of lines BY, CZ, and AA', CZ, AX, and BB', AX, BY, and CC' in (a) are concurrent, and since, by hypothesis, the three sets BY, CZ, and AA_1, CZ, AX, and BB_1, AX, BY, and CC_1 in (b) are concurrent, therefore three pairs of lines AA' and AA_1, BB' and BB_1, CC' and CC_1 coincide, and therefore &c. And, if O be given, then since, by the relations of Cor. 2°, the three sets of lines BO, CO, and B_1C_1, CO, AO, and C_1A_1, AO, BO, and A_1B_1 in (b) intersect collinearly with the opposite sides of ABC, and since, by hypothesis, the three sets BO, CO, and $B'C'$, CO, AO, and $C'A'$, AO, BO, and $A'B'$ in (a) do the same, therefore the three points X, Y, Z are the same for both, and therefore &c.

COR. 4°. Given, with the triangle ABC, either the point O or the line I containing the three points X, Y, Z, which in the modern geometry of the triangle are intimately connected with each other, and distinguished by correlative names expressive of their relation to each other and the triangle, the other may be immediately determined by mere linear constructions based on the above properties a and b, as follows—

For, the triangle ABC being given, the point O gives the three lines OA, OB, OC, they the three points A', B', C', they the three lines $B'C'$, $C'A'$, $A'B'$, they the three points X, Y, Z, and they finally the line I, by property (a); and conversely,

the triangle ABC being given, the line I gives the three points X, Y, Z, they the three lines AX, BY, CZ, they the three points A_1, B_1, C_1, they the three lines AA_1, BB_1, CC_1, and they finally the point O, by property (b).

Cor. 5°. The point O, or line I—and with either of course the other—being given, if from the original triangle ABC two series of triangles $A'B'C'$, $A''B''C''$, $A'''B'''C'''$, &c., and $A_{,}B_{,}C_{,}$, $A_{,,}B_{,,}C_{,,}$, $A_{,,,}B_{,,,}C_{,,,}$, &c. be derived by the continued repetition of the two inverse constructions indicated in the statements of the properties a and b; applied first to the original triangle itself ABC, as in the figure, producing the two first derivatives $A'B'C'$ and $A_{,}B_{,}C_{,}$, then to each of them, in the same manner, producing the two second derivatives $A''B''C''$ and $A_{,,}B_{,,}C_{,,}$, then to each of them again, in the same manner, producing the two third derivations $A'''B'''C'''$ and $A_{,,,}B_{,,,}C_{,,,}$, and so on to infinity; the two series of triangles thus derived from ABC, by the directing agency of O and I, would form evidently, through the connecting link of the original, one continuous, and in both directions unlimited, system of connected triangles, each inscribed to one and exscribed to the other of the two between which it lies; their three systems of corresponding sides passing in different directions through the same three points X, Y, Z on the line I; their three systems of corresponding vertices lying in different positions on the same three lines OA, OB, OC through the point O; and the point and line O and I having to each and all of them, individually and collectively, the same relations as to the original ABC.

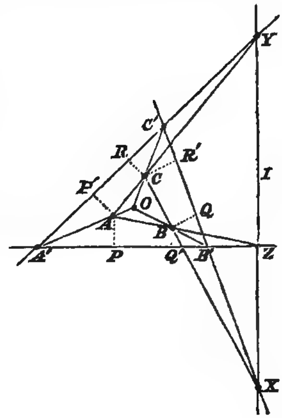
In the particular case of the line I being at infinity, the triangles constituting the system would evidently be all similar, alternately similarly and oppositely placed, and having all the point O for common centre of similitude, (42).

140. The next Example, again, is given separately from the importance of the property as the basis of the theory of perspective, or homology, as it is termed by the French writers, in the geometry of plane figures.

For two triangles of any nature whose vertices and sides correspond in pairs, when the three pairs of corresponding vertices connect concurrently the three pairs of corresponding sides intersect

collinearly, and conversely, when the three pairs of corresponding sides intersect collinearly the three pairs of corresponding vertices connect concurrently.

For, if ABC and $A'B'C'$ be *any* two triangles whose vertices and sides correspond in pairs, AA', BB', CC', the three connectors of corresponding pairs of vertices, and X, Y, Z the three intersections of corresponding pairs of sides; from the vertices ABC of either letting fall pairs of perpendiculars AP and AP', BQ and BQ', CR and CR' upon the pairs of sides about the corresponding vertices A', B', C' of the other, since then, in all cases,



$$BQ : CR' = BX : CX,$$
$$CR : AP' = CY : AY,$$
$$AP : BQ' = AZ : BZ;$$

therefore, in all cases, the two compounds

$$(BQ : CR').(CR : AP').(AP : BQ'),$$

or $$(AP : AP').(BQ : BQ').(CR : CR'),$$

and $$(BX : CX).(CY : AY).(AZ : BZ),$$

are equal in magnitude and similar in sign, and therefore when either $= +1$ so is the other also; but when the former $= +1$ the three lines AA', BB', CC' through the vertices of A', B', C' are concurrent, and conversely, (135), and when the latter $= +1$ the three points X, Y, Z on the sides of ABC are collinear, and conversely, (134, *a*.), and therefore &c. Of course when either equivalent $= -1$ so too is the other also, but the general property resulting from the circumstance, though equally obvious, is not equally important in that case.

As both parts, *a* and *b*, of the property of the preceding Article are evidently included in the above as particular cases, the former, therefore, though independently established in the text, are not really independent, but are merely converse properties; which is evident also from the obvious consideration, adverted to in Cor. 5°, that the two derived triangles $A'B'C'$ and $A_1B_1C_1$,

see figure to the preceding Article, are related each to the original ABC as the original to the other.

141. From the above the following important extension of itself may be readily inferred, viz.—

For two geometrical figures of any kind, F and F', which are of such a nature that, to every point of one corresponds a point of the other, to every line of one a line of the other, to every point of intersection of two lines of one the point of intersection of the two corresponding lines of the other, and to every line of connection of two points of one the line of connection of the two corresponding points of the other; when the several pairs of corresponding points connect concurrently the several pairs of corresponding lines intersect collinearly, and conversely, when the several pairs of corresponding lines intersect collinearly the several pairs of corresponding points connect concurrently.

For, if, in the former case, L and L', M and M', N and N' be any three pairs of corresponding lines, and therefore, by the assumed connections, MN and $M'N'$, NL and $N'L'$, LM and $L'M'$ three pairs of corresponding points, of the figures; since then, by hypothesis, the three latter connect concurrently, therefore, by the above, the three former intersect collinearly; and the property being thus true for every three is therefore true for all pairs of corresponding lines, and therefore &c.; and if, in the latter case, P and P', Q and Q', R and R' be any three pairs of corresponding points, and therefore, by the assumed connections, QR and $Q'R'$, RP and $R'P'$, PQ and $P'Q'$ three pairs of corresponding lines, of the figures; since then, by hypothesis, the three latter intersect collinearly, therefore, by the above, the three former connect concurrently; and the property being thus true for every three is therefore true for all pairs of corresponding points, and therefore &c.

Every two triangles, or figures of any nature related as above to each other, when so relatively situated that their several pairs of corresponding points connect concurrently and their several pairs of corresponding lines intersect collinearly, are said to be *in perspective*, or, as the French writers term it, *in homology* with each other; and, in the same case, the point of concurrence O of the several concurrent connectors, and the

line of collinearity I of the several pairs of collinear intersections, either or both of which may be at infinity, are termed respectively *the centre* and *the axis* of perspective or homology; the meaning and origin of the terms are obvious.

142. Two *similar* figures F and F', both right or left, whatever be their nature, when placed either in similar or in opposite positions with respect to each other (41), furnish the most obvious as well as the simplest example of figures in perspective; for, their several pairs of homologous points P and P' connect concurrently through their *centre of similitude* (42), which therefore in their case is the centre of perspective; and, their several pairs of homologous lines L and L', being parallels, intersect collinearly on *the line at infinity* (136, 1°.), which therefore in their case is the axis of perspective.

Conversely, when the axis of perspective of two figures F and F' in perspective is at infinity, the figures themselves, whatever be their nature, are similar, both right or left, and either similarly or oppositely placed; for, as their several pairs of corresponding lines L and L' intersect at infinity, they are parallel, and, as their several pairs of corresponding points P and P' connect through the centre of perspective, that point satisfies for the figures the conditions of similitude (32), and therefore &c. When, in addition, the centre of perspective also is at infinity, the ratio of similitude being then $= +1$, the figures are not only similar in form, and similarly placed in position, but also equal in magnitude.

143. Two figures F and F' composed of pairs of corresponding points P and P', Q and Q', R and R', &c., connecting by parallel lines all cut in the same ratio by the same line I, furnish another obvious example of figures in perspective, the line of section being the axis, and the point at infinity in the direction of the parallels the centre, of perspective; for perpendicular section generally, every two such figures are said also to be *refractions*, and in the particular case when the ratio of section $= -1$, as already mentioned in (50), to be *reflections* of each other, with respect to the line or axis of section; the general, like the particular, name having been introduced for convenience into Geometry from the science of Optics.

Conversely, when the centre of perspective of two figures F and F' in perspective is at infinity, the figures themselves, whatever be their nature, are connected with each other by the preceding relation; for, as every two connectors PP' and QQ' of their pairs of corresponding points intersect at infinity, they are parallel, and, as the two corresponding lines PQ and $P'Q'$ connecting their extremities intersect on the axis of perspective, they are divided by that line in the same ratio, and therefore &c. When, in addition, the axis of perspective also is at infinity, the ratio of section being then $= + 1$, the figures, which for that ratio would necessarily coincide were the axis not at infinity, are, as already noticed in the preceding article, exact duplicates in form, magnitude, and direction, and merely separated from each other in absolute position by an interval of finite magnitude.

144. Two figures F and F' may be of such a nature as to form, and so circumstanced as to position, that a correspondence between their points and lines in pairs, satisfying the conditions of perspective, may exist in more ways than one. Two similar figures, for instance, of such a form as to be susceptible simultaneously of similar and opposite positions by different ways of correspondence (35), are of such a character, and are accordingly not only in perspective but *doubly* in perspective when in any positions of similitude or opposition, the two centres of similitude, external and internal, being the centres of the two perspectives, and the line at infinity the common axis of both.

Two *circles* being similar figures which, however situated, are *always* at once in similar and opposite positions with respect to each other, are therefore always in perspective for each centre of similitude; but, as we shall see in another chapter, they possess moreover the additional property of being not only in perspective but *doubly* in perspective for *each* centre of similitude, the line at infinity being the common axis for two of the perspectives, and another line at a finite distance the common axis for the other two.

145. In the perspective of two rows of points on different axes or of pencils of lines through different vertices, already

alluded to in (130), an exceptional peculiarity presents itself, which, if not attended to, might cause embarrassment in the applications of the general theory to their particular cases; while the centre of perspective in the case of the rows, and the axis of perspective in the case of the pencils, is unique and determinate (130), the axis in the former case, and the centre in the latter, is indeterminate; every line concurrent with the axes of the rows in the former case, and every point collinear with the centres of the pencils in the latter case, being indifferently an axis of perspective in the one case, and a centre of perspective in the other. All such cases however are exceptional, figures in perspective having in general but a single centre and a single axis of perspective, both generally at a finite distance, but either or both of which may be at infinity.

146. The following are a few consequences from the fundamental theorem of the preceding article (140) respecting triangles in perspective—

a. When three pairs of points P and P', Q and Q', R and R' connect concurrently, the six centres of perspective X and X', Y and Y', Z and Z' of the three pairs of segments QQ' and RR', RR' and PP', PP' and QQ' they determine (131), *lie in four groups of three XYZ, $Y'Z'X$, $Z'X'Y$, $X'Y'Z$ on four lines; each pair of corresponding centres thus constituting a pair of opposite intersections of the same tetragram* (106).

a'. When three pairs of lines L and L', M and M', N and N' intersect collinearly, the six axes of perspective U and U', V and V', W and W' of the three pairs of angles MM' and NN', NN' and LL', LL' and MM' they determine (131), *pass in four groups of three UVW, $V'W'U$, $W'U'V$, $U'V'W$ through four points; each pair of corresponding axes thus constituting a pair of opposite connections of the same tetrastigm* (106).

For, in the former case, the directions of the three segments PP', QQ', RR' being by hypothesis concurrent, the three pairs of points P and P', Q and Q', R and R' determine therefore four pairs of triangles PQR and $P'Q'R'$, QRP' and $Q'R'P$, RPQ' and $R'P'Q$, PQR' and $P'Q'R$, whose pairs of corresponding sides by (140) intersect collinearly at the six centers of perspective of the three segments, viz. QR and $Q'R'$ at X,

RP and $R'P'$ at Y, PQ and $P'Q'$ at Z, QR' and $Q'R$ at X', RP' and $R'P$ at Y', PQ' and $P'Q$ at Z', and therefore &c.; and, in the latter case, the vertices of the three angles LL', MM', NN' being by hypothesis collinear, the three pairs of lines L and L', M and M', N and N' determine therefore four pairs of triangles LMN and $L'M'N'$, MNL' and $M'N'L$, NLM' and $N'L'M$, LMN' and $L'M'N$, whose pairs of corresponding vertices, by (140), connect concurrently by the six axes of perspective of the three angles, viz., MN and $M'N'$ by U, NL and $N'L'$ by V, LM and $L'M'$ by W, MN' and $M'N$ by U', NL' and $N'L$ by V', LM' and $L'M$ by W', and therefore &c.

b. When three triads of points P, Q, R; P', Q', R'; P'', Q'', R'' determine three triangles whose sides pass concurrently through three collinear points, the three conjugate triads P, P', P''; Q, Q', Q''; R, R', R'' also determine three triangles whose sides pass concurrently through three collinear points.

b'. When three triads of lines L, M, N; L', M', N'; L'', M'', N'' determine three triangles whose vertices lie collinearly on three concurrent lines, the three conjugate triads L, L', L''; M, M', M''; N, N', N'' also determine three triangles whose vertices lie collinearly on three concurrent lines.

For, in the former case, if L, M, N; L', M', N'; L'', M'', N'' be the three triads of sides of the three original, and U, U', U''; V, V', V''; W, W', W'' those of the three conjugate triangles; then, since by hypothesis the three triads of points $L'L''$, $M'M''$, $N'N''$; $L''L$, $M''M$, $N''N$; LL', MM', NN' are collinear, therefore by (140) the three triads of lines U, V, W; U', V', W'; U'', V'', W'' are concurrent; and again, since by hypothesis the three triads of lines L, L', L''; M, M', M''; N, N', N'' are concurrent, therefore, by (140), the three triads of points VW, $V'W'$, $V''W''$; WU, $W'U'$, $W''U''$; UV, $U'V'$, $U''V''$ are collinear, and therefore &c. And in the latter case, if P, Q, R; P', Q', R'; P'', Q'', R'' be the three triads of vertices of the three original, and X, X', X''; Y, Y', Y''; Z, Z', Z'' those of the three conjugate triangles; then, since by hypothesis the three triads of lines $P'P''$, $Q'Q''$, $R'R''$; $P''P$, $Q''Q$, $R''R$; PP', QQ', RR' are concurrent, therefore, by (140), the three triads of points X, Y, Z; X', Y', Z'; X'', Y'', Z'' are collinear; and again, since by hypothesis the three triads of points P, P', P''; Q, Q', Q'';

R, R', R'' are collinear, therefore, by (140), the three triads of lines YZ, $Y'Z'$, $Y''Z''$; ZX, $Z'X'$, $Z''X''$; XY, $X'Y'$, $X''Y''$ are concurrent, and therefore &c.

c. When three figures of any kind F, F', F'', in prespective two and two, have a common axis of perspective, the three centers of perspective of the three pairs they determine are collinear.

c'. When three figures of any kind F, F', F'', in perspective two and two, have a common centre of perspective, the three axes of perspective of the three pairs they determine are concurrent.

For, in the former case, if P, Q, R; P', Q', R'; P'', Q'', R'' be any three triads of corresponding points of the three figures; then, since by hypothesis the three triads of lines QR, $Q'R'$, $Q''R''$; RP, $R'P'$, $R''P''$; PQ, $P'Q'$, $P''Q''$ pass concurrently through three collinear points, therefore, by the preceding (b), the three triads of lines $P'P''$, $Q'Q''$, $R'R''$; $P''P$, $Q''Q$, $R''R$; PP', QQ', RR' also pass concurrently through three collinear points, and therefore &c. And, in the latter case, if L, M, N; L', M', N'; L'', M'', N'' be any three triads of corresponding lines of the three figures; then, since by hypothesis the three triads of points MN, $M'N'$, $M''N''$; NL, $N'L'$, $N''L''$; LM, $L'M'$, $L''M''$ lie collinearly on three concurrent lines, therefore, by the preceding (b'), the three triads of points $L'L''$, $M'M''$, $N'N''$; $L''L$, $M''M$, $N''N$; LL', MM', NN' also lie collinearly on three concurrent lines, and therefore &c.

These two latter properties the reader may easily verify, *a priori*, for the particular cases when the common axis in the former case and the common centre in the latter case is at infinity.

147. *When two triangles ABC and $A'B'C'$, whose vertices and sides correspond in pairs, are in perspective.*

a. The sides of each intersect with the non-corresponding pairs of sides of the other so as to fulfil (see fig.)
for ABC the general relation

$$\frac{BY.BZ'}{CY.CZ'}\cdot\frac{CZ.CX'}{AZ.AX'}\cdot\frac{AX.AY'}{BX.BY'}=+1,$$

for $A'B'C'$ the corresponding relation

$$\frac{B'Y'.B'Z}{C'Y'.C'Z}\cdot\frac{C'Z'.C'X}{A'Z'.A'X}\cdot\frac{A'X'.A'Y}{B'X'.B'Y}=+1.$$

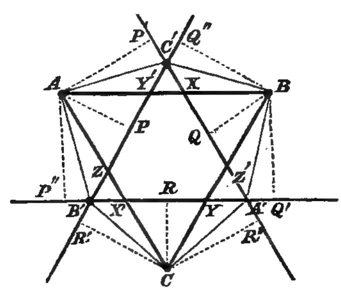
b. The vertices of each connect with the non-corresponding pairs of vertices of the other so as to fulfil (see fig.) *for ABC the general relation*

$$\frac{\sin BAB'.\sin BAC'}{\sin CAB'.\sin CAC'}\cdot\frac{\sin CBC'.\sin CBA'}{\sin ABC'.\sin ABA'}\cdot\frac{\sin ACA'.\sin ACB'}{\sin BCA'.\sin BCB'}=+1,$$

for A'B'C' the corresponding relation

$$\frac{\sin B'A'B.\sin B'A'C}{\sin C'A'B.\sin C'A'C}\cdot\frac{\sin C'B'C.\sin C'B'A}{\sin A'B'C.\sin A'B'A}\cdot\frac{\sin A'C'A.\sin A'C'B}{\sin B'C'A.\sin B'C'B}=+1,$$

and conversely, when two triangles ABC and A'B'C', whose vertices and sides correspond in pairs, are such that the sides of one intersect with the non-corresponding pairs of sides of the other so as to fulfil either relation (*a*), *or that the vertices of one connect with the non-corresponding pairs of vertices of the other so as to fulfil either relation* (*b*), *they are in perspective.*



For, from the three vertices of either triangle ABC, letting fall the three triads of perpendiculars AP, AP', AP''; BQ, BQ', BQ''; CR, CR', CR'' upon the three sides, corresponding and non-corresponding, of the other $A'B'C'$; then, since, in the case of (*a*), by pairs of similar right-angled triangles,

$$BY : CY = BQ' : CR \text{ and } BZ' : CZ' = BQ : CR'',$$
$$CZ : AZ = CR' : AP \text{ and } CX' : AX' = CR : AP'',$$
$$AX : BX = AP' : BQ \text{ and } AY' : BY' = AP : BQ'';$$

and since, in the case of (*b*), by (61), directly

$$\sin B'A'B : \sin C'A'B = BQ' : BQ$$
and $$\sin B'A'C : \sin C'A'C = CR : CR'',$$
$$\sin C'B'C : \sin A'B'C = CR' : CR$$
and $$\sin C'B'A : \sin A'B'A = AP : AP'',$$
$$\sin A'C'A : \sin B'C'A = AP' : AP$$
and $$\sin A'C'B : \sin B'C'B = BQ : BQ'',$$

therefore the left-hand numbers of the first relation (a) and of the second relation (b) are always equal in magnitude and sign to the compound

$$(BQ' : CR'').(CR' : AP'').(AP' : BQ''),$$

or which is the same thing to the compound

$$(AP' : AP'').(BQ' : BQ'').(CR' : CR''),$$

which, by (135), $= +1$ when the triangles are in perspective, and conversely, and therefore &c.

By simply interchanging the two triangles in the preceding construction and demonstrations, the second relation (a), which is for $A'B'C'$ what the first is for ABC, and the first relation (b), which is for ABC what the second is for $A'B'C'$, result of course in the same manner.

148. With an important example of the application of each of the preceding criteria of perspective between triangles, whose vertices and sides correspond in pairs, we shall conclude the present chapter.

Example of criterion (a).—*In every hexagon inscribed in a circle the two triangles determined by the two sets of alternate sides are in perspective, opposite sides in the hexagon being corresponding sides in the perspective.*

For, supposing in the figure of the preceding article the six vertices X and X', Y and Y', Z and Z' of the hexagon $YZ'XY'ZX'$ determined by the six sides of the two triangles ABC and $A'B'C'$ to be six points on a circle; then, since Euc. III. 35, 36,

$$AX.AY' = AZ.AX',\ BY.BZ' = BX.BY',\ CZ.CX' = CY.CZ',$$

therefore relation (a) for the triangle ABC is satisfied in the simplest manner of which it is susceptible, and therefore &c.

This is the celebrated *Theorem of Pascal* respecting a hexagon inscribed in a circle, and accordingly the centre and axis of the perspective in this case are often spoken of as *the Pascal point and line of the hexagon.*

Example of criterion (b).—*In every hexagon circumscribed to a circle the two triangles determined by the two sets of alternate vertices are in perspective, opposite vertices in the hexagon being corresponding vertices in the perspective.*

For, supposing in the same figure the six sides BC' and $B'C'$, CA' and $C'A$, AB' and $A'B$ of the hexagon $BC'AB'CA$, determined by the six vertices of the two triangles ABC and $A'B'C'$ to be six tangents to a circle; then, if a, b, c be the lengths of the three chords intercepted by the circle on the three sides BC, CA, AB of either triangle ABC, since, by (66, Cor. 2°),

$$\sin BAB'.\sin BAC' : \sin CAB'.\sin CAC' = c^2 : b^2,$$

$$\sin CBC'.\sin CBA' : \sin ABC'.\sin ABA' = a^2 : c^2,$$

$$\sin ACA'.\sin ACB' : \sin BCA'.\sin BCB' = b^2 : a^2,$$

therefore relation (b) is satisfied for the triangle ABC, and therefore &c.

This is the celebrated *Theorem of Brianchon* respecting a hexagon circumscribed to a circle, and accordingly the centre and axis of the perspective in this case are often spoken of as *the Brianchon point and line of the hexagon.*

If of the two triangles ABC and $A'B'C'$ one be either inscribed or exscribed to the other, and the latter therefore either exscribed or inscribed to the former, the circle in either of the above properties would manifestly pass through the three vertices of the inscribed and there touch the three sides of the exscribed triangle, and the two properties of triangles circumscribed and inscribed to a circle, given in examples 3° and 4°, Art. 137, would follow at once as particular cases from either of the above.

CHAPTER IX.

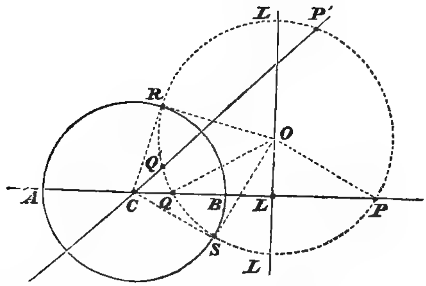
THEORY OF INVERSE POINTS WITH RESPECT TO A CIRCLE.

149. Every two points P and Q on any diameter of a circle, the rectangle $CP.CQ$ under whose distances from the centre C is equal in magnitude and sign to the square of the radius CR, are said to be *inverse points* with respect to the circle.

From the mere definition of inverse points it is evident that: 1°. When the radius of the circle is real they always lie at the same side of the centre and at opposite sides of the circumference, and coincide on the latter when their common distance from the former is equal to the radius; 2°. When the radius is imaginary they always lie at opposite sides of the centre, never coincide, and are at their least distance asunder when equidistant from the centre; 3°. Whether the radius be real or imaginary, as one recedes from, the other approaches to the centre, and conversely, and when one is at infinity in any direction the other is at the centre, and conversely; 4°. In the extreme case when the radius is evanescent, and the circle therefore a point, one is always at the point and the other any where indifferently; 5°. In the other extreme case when the radius is infinite, and the part of the circle not at infinity therefore a line, they are simply reflexions of each other with respect to the line (50). Of these particulars the last, less evident than the others, will appear more fully from the following general property of inverse points.

150. *If P and Q be any pair of inverse points with respect to a circle of any nature, A and B the two extremities, real or imaginary, of the diameter on which they lie, and C the centre of the circle, then always*

$$AP^2 : AQ^2 = BP^2 : BQ^2 = CP : CQ.$$



For, since, by hypothesis, $CP.CQ = CR^2$, therefore

$$CP : CR = CR : CQ = CP \pm CR : CR \pm CQ,$$

and therefore $(CP \pm CR)^2 : (CR \pm CQ)^2 = CP : CQ$, but

$$CP + CR = AP,\ CQ + CR = AQ,\ CP - CR = BP,\ CQ - CR = BQ;$$

and therefore &c.

Hence, in the particular case when C is at infinity, that is, when the part of the circle not at infinity with it is a line at a finite distance; since then $CP : CQ = 1$, therefore, by the above, $AP^2 : AQ^2 = 1$ and $BP^2 : BQ^2 = 1$, or the two points A and B are the two points of bisection, external and internal, of the segment PQ, and therefore, as stated in the preceding Article, *the two points P and Q are in that case reflexions of each other with respect to the line into which the part of the circle not at infinity then opens out.*

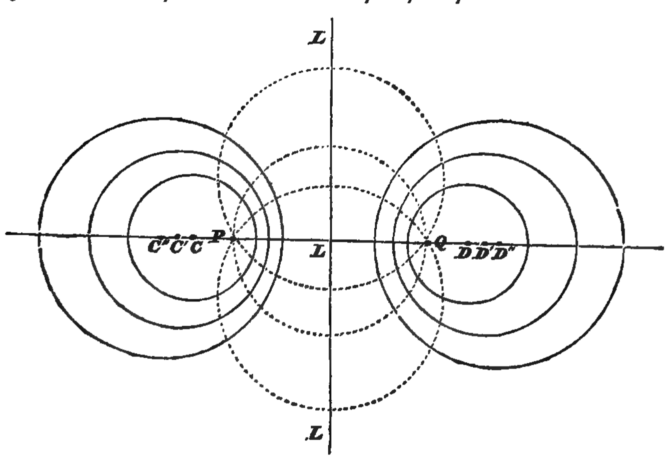
In the Geometry of the Circle, upon which we are now formally entering, the reader will find, as he proceeds, that universally, as above, when the centre of a circle goes off to infinity without carrying the entire circle with it, the line at a finite distance, into which the figure in its limiting form for the extreme magnitude of its radius $= \infty$ then opens out (18), is in reality but *part* of the entire circle; *the line at infinity* (136) being invariably the remaining part, and possessing, in combination with the line not at infinity, all the properties of the complete circle in the general case; instances confirmatory of this will appear in numbers in the sequel, and though to avoid circumlocution we shall continue generally to speak, as we have hitherto done, of a circle becoming a line when its centre goes to infinity leaving itself behind, the circumstance that the line

at infinity is always to be associated with the line not at infinity as part of the entire circle must never, in such cases, be lost sight of whenever it may be necessary, as it often is, to take it into account.

151. Whatever be the nature of the circle, the inverse Q of every point P, not the centre C, is evidently unique and determinate, being on the line CP connecting two known points C and P, and at a distance CQ from one of them C of known magnitude and sign; of the centre itself, however, the inverse, being on the line connecting two coincident points, is indeterminate, *any point at infinity* when the radius is finite, and *any point whatever* in the particular case when it is evanescent, satisfying evidently the conditions that determine it.

When two points P and Q are such that one P is the inverse of the other Q with respect to any circle, the latter Q is, of course, reciprocally, the inverse of the former P with respect to the same circle.

152. As every circle has an infinite number of pairs of inverse points P and Q, P' and Q', P'' and Q'', &c., whose lines of connection all pass through its centre C, and for which the several rectangles $CP.CQ$, $CP'.CQ'$, $CP''.CQ''$, &c. are all equal in magnitude and sign to the square of its radius CR; so conversely, every two points P and Q have an infinite number of circles to which they are inverse, whose centres C, C', C'', &c. all lie on their



line of connection PQ, and the squares of whose radii CR, $C'R'$, $C''R''$, &c. are severally equal in magnitude and sign to the corresponding rectangles $CP.CQ$, $C'P.C'Q$, $C''P.C''Q$, &c.; every such circle is said, for a reason that will appear in another chapter, to be *coaxal* with the two points P and Q, and its radius CR is evidently real or imaginary according as its centre C is external or internal to the segment PQ, evanescent when C coincides with either point P or Q, and infinite when C is at infinity, in which case the line into which the part of the circle not at infinity then opens out is (150) the axis of reflexion L of the two points P and Q.

Every two circles belonging to such a system being evidently equal in magnitude when their centres C and D, C' and D', C'' and D'', &c. are equidistant in opposite directions from the middle point of PQ, the entire system consists therefore of two similar and opposite groups, symmetrically disposed in equal and opposite pairs, reflexions of each other with respect to the axis of reflexion of P and Q, by and through which, in combination with the line at infinity, the circles of one group are separated from and pass into those of the other; each circle of each group enclosing all within and being enclosed by all without itself; and each point P and Q being the nucleus round which the circles of its own group are eccentrically disposed, and the evanescent limit through which they pass from real to imaginary, and conversely.

In the particular case when the two points P and Q coincide, the circles of the system are all real, the range of centres PQ for which they are imaginary in the general case being then evanescent. In this, the only case in which any two circles of the entire system have a common point or any two of the same group a common tangent, they evidently all pass through the point of coincidence $P=Q$, and all touch at that point the line L passing through it perpendicular to their line of centres; and all the other particulars respecting their distribution, as above stated for the general case, are obvious, and have been already stated in Art. 18.

153. In connection with the subject of the preceding Article the following problem not unfrequently presents itself:

Given two pairs of points P and Q, P' and Q' on the same line, to determine the centre C and radius CR of the circle coaxal with both.

To solve which, since, by the preceding,

$$CP.CQ = CP'.CQ' = CR^2,$$

therefore, assuming arbitrarily any point M not on the line, describing through it the two circles PMQ and $P'MQ'$, and drawing their chord of intersection MN intersecting the given line at the point C; the circle round C as centre, the square of whose radius CR is equal in magnitude and sign to the rectangle $CM.CN$, is evidently that required. For (Euc. III. 35, 36)

$$CP.CQ = CP'.CQ' = CM.CN = CR^2,$$

and therefore &c.

The circle thus determined, though its centre C is always real, is itself imaginary when the two points P and Q alternate with the two P' and Q' in the order of their occurrence on their common axis; this is evident from the obvious circumstance that the rectangle $CM.CN$ is then necessarily negative; in every other case however it is positive, and the circle is therefore real.

In the particular case when the two intercepted segments PQ and $P'Q'$ have a common middle point, the centre C, determined as above, being then at infinity, the part of the circle itself not at infinity opens out, as it ought, into the common axis of reflexion of the two pairs of points P and Q, P' and Q', see (150).

154. Any two segments of the same diameter of a circle, which are such that the extremities of one are the inverses of the extremities of the other with respect to the circle, are termed *inverse segments* with respect to the circle; thus, if PP' be the segment intercepted between any two points P and P' on the same diameter of a circle, and QQ' that intercepted between the two inverse points Q and Q', the two segments PP' and QQ' are inverse segments with respect to the circle.

Since, from the interchangeability of inverse points (151), every two pairs of inverse points P and Q, P' and Q' on the same diameter of a circle, determine evidently two different pairs of inverse segments PP' and QQ', PQ' and QP', hence

connected with every pair of inverse segments PP' and QQ' with respect to any circle, there exists always a conjugate pair PQ' and QP' with respect to the same circle.

Again, as every two segments PQ and $P'Q'$ of the same line thus determine two different pairs of segments PP' and QQ', PQ' and QP' inverse to the unique circle coaxal with themselves (153), so conversely, they determine two different circles with respect to which they are themselves inverse segments, one that coaxal with the two PP' and QQ', and the other that coaxal with the two PQ' and QP' (153).

Hence the useful problem, *given two segments PQ and $P'Q'$ of the same line, to determine the two circles with respect to which they are inverse segments*, is reduced to that of the preceding Article (153), viz. to determine the two circles which are coaxal, one with the two segments PP' and QQ', and the other with the two PQ' and QP', and which, from the construction given in that Article, are easily seen to be both real in the case when the extremities of the two given segments PQ and $P'Q'$ alternate with each other in the order of their occurrence on their common axis, and to be one real and one imaginary in either of the two cases when they do not.

155. *Every two points and their two inverses with respect to the same circle lie in a circle.*

For, if (fig., Art. 150) P and P' be the two points, Q and Q' their two inverses, and C the centre of the circle; then since, by the definition of inverse points, $CP.CQ = CP'.CQ'$, each being = the square of the radius of the circle, therefore &c.

Conversely, *every circle passing through a pair of inverse points with respect to another circle determines a pair of inverse points on every diameter of the other.*

For, if P and Q, P' and Q' (same figure) be the two pairs of points in which any circle intersects any two diameters of any other circle, and C the centre of the latter; then, since $CP.CQ = CP'.CQ'$, if either rectangle = the square of either radius, so is the other.

Cor. 1°. It is evident from the above that if the same circle pass through a pair of inverse points with respect to one circle, and also through a pair of inverse points with respect to an-

other circle, it cuts the diameter common to both in a pair of inverse points common to both.

Cor. 2°. The preceding furnishes an obvious solution of the problem, "*to determine on the common diameter of two given circles the two points, real or imaginary, inverse to both.*" For, assuming arbitrarily any point P, and describing the circle passing through it and through its two inverses Q and R with respect to the two circles; the circle PQR thus described intersects, by the preceding, the common diameter in the two points required.

The two points E and F thus determined are imaginary when the two circles intersect, real when they do not, and coincident at the point of contact when they touch. See Art. 152.

156. *Every circle passing through a pair of inverse points with respect to another circle is orthogonal to the other.* (22).

For, if C (fig., Art. 150) be the centre of any circle, P and Q any pair of inverse points with respect to it, and R either point in which any circle through P and Q intersects it; since then by hypothesis $CP.CQ = CR^2$, therefore CR, a radius of one circle, is a tangent to the other, and therefore &c. (22).

Conversely, *every circle orthogonal to another determines pairs of inverse points on all diameters of the other.*

For, if C (same fig.) be the centre of either circle, P and Q the two points in which any line through it meets the other, and R either point of intersection of the two; then since the radius CR of the former is, by hypothesis, a tangent to the latter, therefore $CP.CQ = CR^2$, and therefore &c. (22).

Cor. 1°. It is evident from the above that *every circle passing through the common pair of inverse points with respect to two others* (155, Cor. 2°) *is orthogonal to both*, and conversely that, *every circle orthogonal to two others passes through their common pair of inverse points.*

Cor. 2°. It is also evident from the same that *all the circles of a system having a common pair of inverse points* (see the undotted circles of fig., Art. 152) *are cut orthogonally by every circle passing through the points*, and, conversely, that *all the circles of a system passing through a pair of common points* (see the dotted circles of same figure) *are cut orthogonally by every circle coaxal with the points.*

COR. 3°. It follows also from the above and from Cor. 1°. that *if a variable circle pass through a fixed point and cut a fixed circle at right angles, or, more generally, if it cut two fixed circles at right angles, the locus of its centre is a line;* for passing through the point and its inverse with respect to the circle in the former case, and through the common pair of inverse points with respect to the two circles in the latter case, its centre in either case describes therefore the axis of reflexion of the two points through which it passes; a more general proof for the second case will be given in another chapter.

COR. 4°. The preceding supply obvious solutions of the three following problems: "*To describe a circle,* 1°. *passing through two given points and cutting a given circle at right angles;* 2°. *passing through a given point and cutting two given circles at right angles;* 3°. *cutting three given circles at right angles.*" For the circle passing through the two points and through the inverse of either with respect to the circle, in the first case; that passing through the point and its two inverses with respect to the two circles, in the second case; and that orthogonal to any one of the three circles, and passing through the common pair of inverse points with respect to the other two, in the third case; is evidently that required; a more general construction for the third case will be given in another chapter.

157. *The two tangents to a circle from any point in the axis of reflexion of any pair of inverse points are equal to the two distances of the point from the inverse points.*

For, if P and Q (fig., Art. 150) be the inverse points, O any point in their axis of reflexion L, and OR and OS the two tangents from O to the circle; since then, by the preceding, the circle round O as centre which passes through P and Q cuts the original circle at right angles, it passes through R and S, and therefore &c.

Conversely, *the locus of a variable point, not at infinity* (15), *the tangents from which to a fixed circle are equal to its distance from a fixed point is a line, the axis of reflexion of the point and its inverse with respect to the circle.*

For, if P (same fig.) be the fixed point, and O any point for

which the two tangents OR and OS to the fixed circle are each equal to the distance OP; since then the circle round O as centre which passes through P passes through R and S, it cuts the fixed circle at right angles, and therefore passes also through Q, the inverse of P with respect to the fixed circle, and therefore &c.

Cor. 1°. It is evident from the first part of the above that when (152) any number of circles have a common pair of inverse points P and Q, tangents to them all from any point in the axis of reflexion L of the two points are equal.

Cor. 2°. The second part of the above supplies of itself obvious solutions of the two following problems:

1°. *To determine the point on a given line or circle, the tangents from which to a given circle shall be equal to its distance from a given point.*

2°. *To determine the point, the tangents from which to two given circles shall be equal to its distances from two given points.*

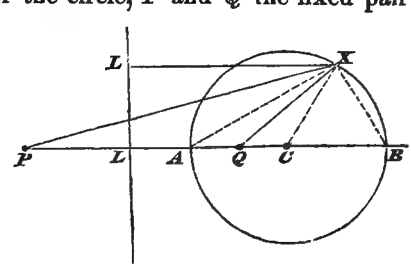
And, by aid of Cor. 2°., Art. (155) of the two following:

1′. *To determine the point on a given line or circle, the tangents from which to two given circles shall be equal.*

2′. *To determine the point, the tangents from which to three given circles shall be equal.*

158. *The squares of the distances of a variable point on a fixed circle from any fixed pair of inverse points have the constant ratio of the distances of the centre from the inverse points.*

For, if C be the centre of the circle, P and Q the fixed pair of inverse points, and X the variable point on the circle; since then, by hypothesis, $CP.CQ = CX^2$, or, which is the same thing, $CP : CX :: CX : CQ$; therefore the triangles PCX and XCQ are similar, and therefore $PX^2 : QX^2 :: PC^2 : CX^2 :: CX^2 : QC^2 :: PC : QC$. The property of Art. (150) is evidently a particular case of this.



Conversely, *the locus of a variable point the distances of which from two fixed points have any constant ratio, is the circle coaxal with the fixed points* (152) *whose centre divides the distance between them in the duplicate of the constant ratio.*

For, if P and Q be the two fixed points, X any position of the variable point, and C the point on PQ for which $CP.CQ = CX^2$; then since, as above, the triangles PCX and XCQ are similar, therefore, as above, $PC : QC :: PX^2 : QX^2$, which being by hypothesis constant, therefore &c.

If while the two points P and Q remain fixed, the constant ratio $PX : QX$ be conceived to vary and pass continuously through all values from 0 to ∞, the locus circle will pass evidently through all the phases of coaxality with P and Q described in (152); see fig. of that article. Commencing with the point P as the nascent limit for the extreme value 0; opening out into the axis of reflexion L of P and Q as the part of the locus not at infinity (150) for the mean value 1; and ending with the point Q as the evanescent limit for the extreme value ∞.

Since for *every* point X *at infinity* the ratio $PX : QX = 1$ (15), the *complete* locus, which for every value of the ratio not $= 1$ is by the above a single unbroken circle in its general form, consists therefore for the particular value of the ratio $= 1$ of *two* lines, viz. the axis of reflexion of P and Q, and the line at infinity (136); this is an instance confirmatory of the general statement made at the close of Art. (150), that when the centre of a circle of infinite radius is at infinity the circle itself breaks up into two lines, one at a finite distance, and the other at infinity.

Cor. 1°. Since from the similarity of the two triangles PCX and QCX in the first part of the above, the two pairs of angles XPC and QXC, XQC and PXC are always similar (24), it follows consequently that—

Of the two lines connecting any point on a circle with any pair of inverse points, the angle determined by either with the radius at the point is similar to that determined by the other with the diameter containing the inverse points.

Cor. 2°. The second part of the above supplies obvious solutions of the two following problems:

1°. *To determine the point on a given line or circle, the ratio of whose distances from two given points shall be given.*

2°. *To determine the point, the ratios of whose distances from three given points shall be given.*

159. *The square of the distance of a variable point on a fixed circle from any fixed point varies as its distance from the axis of reflexion of the point and its inverse with respect to the circle.*

For, if C (figure of last Article) be the centre of the circle, P and Q the fixed point and its inverse, X any position of the variable point on the circle, and XL the perpendicular from X on the axis of reflexion L of P and Q; since then, Euc. II., 5, 6,

$$PX^2 - QX^2 = 2PQ.LX = 2(PC - QC)LX,$$

and since, by the preceding, $PX^2 : QX^2 :: PC : QC$, therefore $PX^2 = 2PC.LX$ and $QX^2 = 2QC.LX$, and therefore &c.

Conversely, *the locus of a variable point the square of whose distance from a fixed point varies as its distance from a fixed line is a circle coaxal with the point and its reflexion with respect to the line* (152).

For, if P (same fig.) be the fixed point, L the fixed line, Q the reflexion of P with respect to L, X any position of the variable point, XL its distance from the fixed line, and C the point on PQ for which $PX^2 = 2PC.LX$; since then, as above,

$$PX^2 - QX^2 = 2PQ.LX = 2(PC - QC)LX,$$

therefore $QX^2 = 2QC.LX$, and therefore $PX^2 : QX^2 :: PC : QC$, from which, since by hypothesis PC is constant, and therefore C fixed, it follows from the preceding that the locus of X is the circle coaxal with P and Q whose centre is C.

If while the point and line P and L remain fixed, the base PC of the variable rectangle $PC.LX$ be conceived to vary and take successively in the direction opposite to that of PQ all values from 0 to ∞, the locus circle will pass evidently through half the system of phases of coaxality with P and Q described in (152); commencing with P as the nascent limit for the extreme value 0, and ending with L as the part not at infinity of the infinite limit for the extreme value ∞. And if then after passing through infinity PC be conceived to change direction and take successively all values from ∞ to PQ, the locus circle

will pass evidently through the remaining half of the same series of phases; commencing with L as the part not at infinity of the infinite limit for the extreme value ∞, and ending with Q as the evanescent limit for the extreme value PQ; after which, changing its nature, it will evidently become and continue imaginary for all lesser values from PQ down to 0.

Cor. The second part of the above supplies obvious solutions of the two following problems:

1°. *To determine the point on a given line or circle, the square of whose distance from a given point shall be equal to the rectangle under a given base and its distance from a given line.*

2°. *To determine the point, the squares of whose distances from two given points shall be equal to the rectangles under two given bases and its distances from two given lines.*

160. *The angle connecting any point on a circle with any pair of inverse points is bisected, internally and externally, by the lines connecting the point with the extremities of the diameter containing the inverse points.*

For, if (same figure as in Art. 158) C be the centre of the circle, P and Q the pair of inverse points, A and B the extremities of the diameter on which they lie, and X any point on the circle; since then, by the first part of (158),

$$PA^2 : QA^2 = PB^2 : QB^2 = PX^2 : QX^2 = PC : QC,$$

therefore (Euc. VI. 3) the angle PXQ is bisected internally and externally by the two lines PA and PB, and therefore &c.

Conversely, *the locus of a variable point the angle connecting which with two of three fixed collinear points is bisected, internally or externally, by the line connecting it with the third, is the circle coaxal with the two which passes through the third.*

For, if P and Q (same figure) be the first and second of the fixed points, A or B the third, C the point on their common line for which $PC : QC = PA^2 : QA^2$ or $PB^2 : QB^2$, and X any position of the variable point; since then, by hypothesis, the angle PXQ is bisected by the line PA or PB, therefore (Euc. VI. 3) $PX^2 : QX^2 = PA^2 : QA^2$ or $PB^2 : QB^2 = PC : QC$, consequently, by the second part of (158), $CX^2 = CP.CQ = CA^2$ or CB^2, and therefore &c.

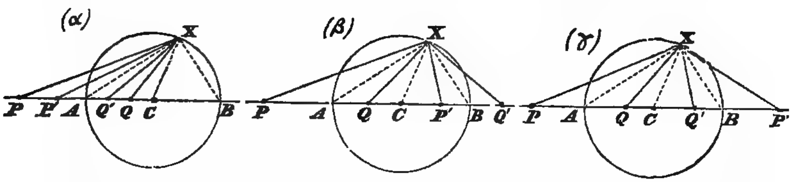
P

COR. The second part of the above supplies obvious solutions of the two following problems:

1°. *To determine the point on a given line or circle, the angle connecting which with two of three given points in a line shall be bisected by the line connecting it with the third.*

2°. *To determine the point, the angles connecting which with the extremities of two given lines shall be bisected by the lines connecting it with two given points on the lines.*

161. *Every two inverse segments of any diameter of a circle* (154) *subtend similar angles* (24) *at every point on the circle.*



For, if PP' and QQ' be the two inverse segments, P and Q, P' and Q' their two pairs of inverse extremities, A and B the extremities of the diameter to which they belong, C the middle point of AB, and X any point on the circle; then since, by the first part of the preceding (160), the two angles PXQ and $P'XQ'$ have the same bisectors XA and XB, therefore the two angles PXP' and QXQ' (and also the two PXQ' and QXP' (154)) are similar, and therefore &c.

Conversely, *the locus of a variable point the angles subtended at which by two fixed coaxal segments are similar, consists of the two circles* (154) *with respect to which the two segments are inverse.*

For, if PP' and QQ' (same figures) be the two segments, and X any position of the variable point; then since, by hypothesis, the two angles PXP' and QXQ' are similar, therefore either the two angles PXQ and $P'XQ'$, or the two PXQ' and QXP', have the same bisectors; in the former case (that of the figures), if C be the middle point of the segment AB intercepted on the axis of the segments by the common bisectors XA and XB, then since, as in the second part of the preceding (160), $CX^2 = CP.CQ = CP'.CQ'$, therefore C and CX are the centre and radius of the circle coaxal with PQ and $P'Q'$ (153), and therefore &c.; and in the latter case (not that of the

figures), if C' be the middle point of the segment $A'B'$ intercepted on the axis of the segments by the common bisectors XA' and XB', then since, for the same reason as before, $C'X^2 = C'P.C'Q' = C'Q.C'P'$, therefore C' and $C'X$ are the centre and radius of the circle coaxal with PQ' and $P'Q$ (153), and therefore &c.

Of the two different circles comprising the above locus, though the first is real for all the three possible modes (82) in which the two segments PP' and QQ' could be disposed on their common axis, as represented in the three figures (α), (β), (γ), the second is real only for the disposition, represented in fig. (β), in which the extremities of one segment alternate with those of the other in the order of their occurrence on their common axis (see 153).

Cor. 1°. From the similarity of the two pairs of angles PXP' and QXQ', PXQ' and QXP' in the first part of the above, it follows immediately from (65), combined with (158), that

$$\frac{PP'.PQ'}{QP'.QQ'} = \frac{PX^2}{QX^2} = \frac{PA^2}{QA^2} = \frac{PB^2}{QB^2} = \frac{PC}{QC},$$

and, of course, for the same reason that

$$\frac{P'P.P'Q}{Q'P.Q'Q} = \frac{P'X^2}{Q'X^2} = \frac{P'A^2}{Q'A^2} = \frac{P'B^2}{Q'B^2} = \frac{P'C}{Q'C},$$

and therefore, generally, that—

The rectangles under the distances of any pair of inverse points from any other pair on the same diameter are as the squares of their distances from each extremity of the diameter, and as their distances from the centre of the circle.

Cor. 2°. The second part of the above supplies obvious solutions of the two following problems:

1°. *To determine the point on a given line or circle, the angles subtended at which by two given coaxal segments shall be similar.*

2°. *To determine the point, the angles subtended at which by three given coaxal segments shall be similar.*

162. *The extremities of any chord of a circle, the centre, and the inverse of any point on the chord, lie in a circle.*

For, if C be the centre of the circle, X and Y the ex-

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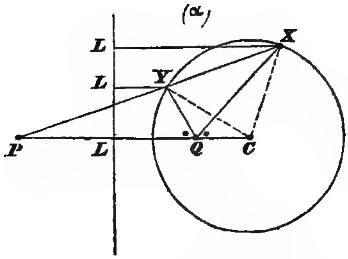
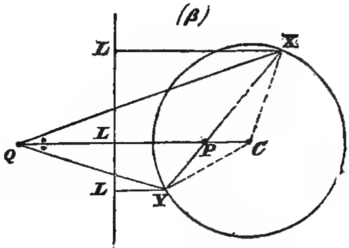




tremities of any chord, P any point, external or internal, on XY, and Q the inverse of P with respect to the circle; since then CX^2 or $CY^2 = CP^2 - PX.PY$, by the isosceles triangle XCY, and $= CP.CQ = CP^2 - PC.PQ$, by the inverse points P and Q, therefore $PX.PY = PC.PQ$, and therefore &c.

Conversely, *every circle passing through the centre of another circle passes through the inverse of every point on its chord of intersection with the other.*

For, if C be the centre of any circle, X and Y its points of intersection with any circle passing through C, P any point, external or internal, on XY, and Q the point in which the circle XCY intersects the line CP; since then $PC.PQ = PX.PY$, therefore $PC^2 - PC.PQ = PC^2 - PX.PY$, that is, $CP.CQ = CX^2$ or CY^2, and therefore &c.

Cor. 1°. From the above, supposing the two points P and Q to remain fixed, and the line and circle XY and XCY to vary simultaneously, it appears that—

If a variable line pass through a fixed point and intersect a fixed circle, the circle passing through the points of intersection and though the centre of the latter passes through a second fixed point, the inverse of the first with respect to the fixed circle.

And conversely, that—

If a variable circle pass through a fixed point and through the centre of a fixed circle, its chord of intersection with the latter passes through a second fixed point, the inverse of the first with respect to the fixed circle.

Cor. 2°. From the same, supposing, conversely, the line XY and circle XCY to remain fixed, and the two points P and Q to vary simultaneously, it appears again that—

If a variable point describe a fixed line, its inverse with respect

to any circle describes the circle determined by the centre of the latter and by its intersections with the fixed line.

And conversely, that—

If a variable point describe a fixed circle, its inverse with respect to any circle through whose centre it passes describes the line determined by the points of intersection of the two circles.

Cor. 3°. In the particular case when P is the middle point of the chord XY, since then CQ is evidently a diameter of the circle XCY, therefore the two angles CXQ and CYQ are both right, and therefore, from the above—

The middle point of any chord of a circle and the intersection of the two tangents at its extremities, and conversely, the intersection of any two tangents to a circle and the middle point of their chord of contact, are inverse points with respect to the circle.

163. *The diameter containing any pair of inverse points with respect to a circle bisects, externally or internally, the angle subtended at either point by any chord of the circle whose direction passes through the other.*

For, if P and Q (figures of last article) be the two points, X and Y the extremities of any chord passing through either of them P, and C the centre of the circle; then, since by (158), $PX^2 : QX^2 = PY^2 : QY^2$, each being $= PC : QC$, therefore, by alternation, $PX^2 : PY^2 = QX^2 : QY^2$, and therefore, Euc. VI. 3, the angle XQY is bisected, externally or internally, by QP; or, since, by (162), the circle XCY passes through Q, as the arc XY is bisected, externally or internally, at C, so is the angle XQY by QC.

Conversely, *if two points on the same diameter of a circle be such that the angle subtended at one of them by any chord of the circle, not perpendicular to the diameter, whose direction passes through the other is bisected by the diameter, they are inverse points with respect to the circle.*

For, if P and Q (same figures as before) be the two points, and XY the chord whose direction passes through one of them P; then, since by hypothesis, the angle XQY is bisected, externally or internally, by QP, therefore, Euc. VI. 3, $PX : QX = PY : QY$, and therefore (158) X and Y are two points on the same circle coaxal with P and Q, which, as its centre lies on the line PQ,

unless in the particular case when XY is perpendicular to PQ, coincides therefore necessarily with the original circle, and therefore &c.; or, if C be the point in which the circle XQY intersects the line PQ, since by hypothesis the angles XQC and YQC are equal or supplemental, therefore the lines CX and CY are equal, and therefore either XY is perpendicular to PQ, or C is the centre of the original circle, in which case (162) $CP.CQ = CX^2$ or CY^2, and therefore &c.

Cor. 1°. It is evident from the above, that *when any number of circles have a common pair of inverse points* (152), *all pairs of opposite segments, intercepted by pairs of them on any line passing through either, subtend similar angles at the other.* For, if XY and $X'Y'$ be the two chords intercepted by any two of them on any line passing through either point P, the two angles XQY and $X'QY'$, subtended by them at the other Q, have the same bisector PQ, and therefore the two pairs of angles XQX' and YQY', XQY' and YQX' are similar.

Cor. 2°. It is also evident from the converse, that *the two centres of perspective of any two parallel chords of a circle are inverse points with respect to the circle.* For, when two chords XY and $X'Y'$ are parallel, the two pairs of opposite lines XX' and YY', XY' and YX' connecting their extremities, two and two, intersect evidently upon, and make equal angles with, the same diameter, and therefore &c.

164. *If a variable chord of a fixed circle turn round a fixed point, the rectangles under the distances of its extremities from the inverse of the point and from the axis of reflexion of the point and its inverse are both constant.*

For, if C (same figures as before) be the centre of the circle, P the fixed point, Q its inverse with respect to the circle, L the axis of reflexion of P and Q, and XY any position of the variable chord turning round P; then, to prove the first, since, by (158),

$$QX^2 : PX^2 = QY^2 : PY^2 = QC : PC,$$

therefore $$QX.QY : PX.PY = QC : PC,$$

and since (Euc. III. 35, 36)

$$PX.PY = PC.PQ = 2PC.PL,$$

therefore $$QX.QY = QC.QP = 2QC.QL,$$

and therefore &c.; and, to prove the second, since, by (159),

$$PX^2 = 2PC.LX \text{ and } PY^2 = 2PC.LY,$$

therefore $$LX.LY = PX^2.PY^2 \div 4PC^2,$$

and since (Euc. III. 35, 36)

$$PX^2.PY^2 = PC^2.PQ^2 = 4PC^2.PL^2,$$

therefore $LX.LY = LP^2$, and therefore &c.

Conversely, *if a variable chord of a fixed circle, turning round one of two fixed points on the same diameter of a circle, be such that the rectangle under the distances of its extremities either from the other or from the axis of reflexion of the two is constant, the two points are inverse points with respect to the circle.*

These are both evident from the direct properties, by taking the two extreme positions of the variable chord, those, viz. in which it coincides with the diameter containing the points, and in which it either intersects that diameter at right angles or touches the circle according as the point round which it turns is external or internal to the latter.

Cor. It follows at once from the above, that *for a system of circles having a common pair of inverse points* (152), *the several rectangles under the distances of the extremities of all chords passing through either from the other are constant, and from the axis of reflexion of both are constant and equal to the square of the semi-segment intercepted between them.*

CHAPTER X.

THEORY OF POLES AND POLARS WITH RESPECT TO A CIRCLE.

165. THE line passing through the inverse of any point with respect to a circle, and intersecting at right angles the diameter containing the point, is termed the *polar* of the point with respect to the circle; and, conversely, the inverse of the foot of the perpendicular from the centre of a circle upon any line is termed the *pole* of the line with respect to the circle.

From the mere definition of a point and line, pole and polar to each other with respect to a circle, it is evident that—In the general case when the radius of the circle is finite, 1°. They lie at the same side or at opposite sides of the centre, according as the circle is real or imaginary; 2°. In either case, as one approaches to or recedes from, the other, conversely, recedes from or approaches to, the centre; 3°. The polar of the centre is the line at infinity, and conversely, the pole of the line at infinity is the centre; 4°. The polar of any point on the circle is the tangent at the point, and conversely, the pole of any tangent to the circle is the point of contact; 5°. The polar of any point at infinity is the diameter perpendicular to the direction of the point, and conversely, the pole of any diameter is the point at infinity in the direction perpendicular to the diameter; 6°. The point of intersection and chord of contact of any two tangents to the circle are pole and polar to each other with respect to the circle (162, Cor. 3°.). In the extreme case when the radius of the circle is evanescent, 1°. Every line, however situated, is a polar of the centre; 2°. Every line, not passing through the centre, is a polar of the centre only; 3°. Every line passing through the centre is a polar, not only of the centre, but of every point indifferently on the orthogonal line passing

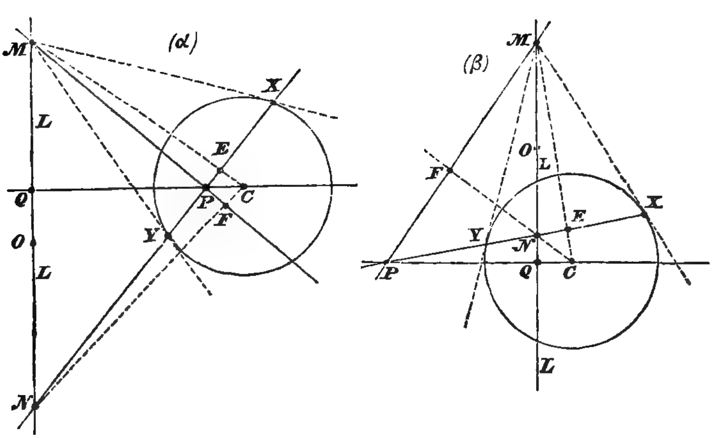
through the centre. And in the extreme case when the radius is infinite, the polar of every point, however situated, is parallel to the line into which the part of the circle not at infinity then opens out, and distant from it at the opposite side by an interval equal to that of the point.

Every two angles being similar whose sides are mutually perpendicular, it is evident also that, whatever be the nature of the circle, the angle subtended at the centre by any two points is similar to that determined by the polars of the points, and conversely, the angle determined by any two lines is similar to that subtended at the centre by the poles of the lines.

In the theory of poles and polars with respect to a circle, the diameter passing through any point is termed *the polar axis of the point*, and the projection of the centre on any line *the polar centre of the line.*

166. Of the various properties of points and lines, pole and polar to each other with respect to a circle, the two following, converse to each other, lead to the greatest number of consequences, and may be regarded as fundamental.

When a line passes through a point, its pole with respect to any circle lies on the polar of the point with respect to the circle; and conversely, *when a point lies on a line, its polar with respect to any circle passes through the pole of the line with respect to the circle.*



To prove which, P and L being the point and line, pole and

polar to each other, and C the centre of the circle; if, in the first case, XY be any line through P, CE the perpendicular from C on XY, and M the point in which CE intersects L; then since, by similar right-angled triangles CEP and CQM, the rectangles $CE.CM$ and $CP.CQ$ are equal, and since, by hypothesis, the latter rectangle $CP.CQ =$ the square of the radius of the circle, therefore the former rectangle $CE.CM$ is also = the square of the radius, and therefore the point M is the pole of the line XY with respect to the circle. And if, in the second case, M be any point on L, MC the line connecting it with C, and PE the line through P perpendicular to MC; then, as before, $CE.CM = CP.CQ =$ square of radius of circle, and therefore the line PE is the polar of the point M with respect to the circle.

Cor. 1°. Since, by the above, the pole of every line passing through P lies on L, and, conversely, the pole of every point lying on L passes through P, it follows consequently that—

If any number of lines of any geometrical figure pass through a point, their poles with respect to any circle lie on a line, the polar of the point with respect to the circle; and conversely, *if any number of points of any geometrical figure lie on a line, their polars with respect to any circle pass through a point, the pole of the line with respect to the circle.*

Cor. 2°. If, in the above, one pole and polar P and L be conceived to remain fixed with the circle, and the other M and XY to vary, it appears again that—

If a variable line turn round a fixed point, its pole with respect to any fixed circle describes a fixed line, the polar of the point with respect to the circle; and conversely, *if a variable point describe a fixed line, its polar with respect to any fixed circle turns round a fixed point, the pole of the line with respect to the circle.*

Cor. 3°. Since when, as in the figures, the points X and Y are real, tangents at them intersect at M, and conversely (162, Cor. 3°), it follows of course, as included in the preceding, that—

If a variable chord of a fixed circle pass through a fixed point, the two tangents at its extremities intersect on a fixed line, the polar of the point; and conversely, *if two variable tangents to a fixed circle intersect on a fixed line, their chord of contact passes through a fixed point, the pole of the line.*

Cor. 4°. It being evident, from the right angle PEC, that as the point M describes the line L its inverse E with respect to the circle describes the circle on PC as diameter, and conversely. Hence, as shewn otherwise for a particular case in (162, Cor. 2°)—

If a point describe a line, its inverse with respect to any circle describes the circle passing oppositely through the centre of the circle and the pole of the line; and conversely, *if a point describe a circle, its inverse with respect to any circle through whose centre it passes describes the line polar with respect to the latter of the point of the former opposite to its centre.*

The above properties, suitably modified, are of course all true in the particular cases when either of the two points P or Q is at infinity, and the other therefore at the centre of the circle.

167. From the fundamental property of the preceding article, it is evident with respect to any circle, that—

The line of connection of any two points is the polar of the point of intersection of the polars of the points; and, reciprocally, the point of intersection of any two lines is the pole of the line of connection of the poles of the lines.

For, by that property, when a line passes through two points its pole lies on the polars of both, and reciprocally, when a point lies on two lines its polar passes through the poles of both, and therefore &c.

The point of intersection and the chord of contact of any two tangents to a circle being pole and polar to each other with respect to the circle (162, Cor. 3°), it follows, of course, as included in the preceding, that *the point of intersection of the two chords of contact and the line of connection of the two points of intersection of any two pairs of tangents to the same circle, are pole and polar to each other with respect to the circle.*

Of the many consequences from the above, which in the modern geometry of the circle are numerous and remarkable, the six next articles contain a few of the most important.

168. *When a triangle is such that two of its vertices and their opposite sides are pole and polar to each other with respect to a circle, the third vertex and its opposite side are pole and polar to each other with respect to the same circle.*

For, since for three points PMN, (see figures of article 166), when P is the pole of MN, and M the pole of PN, then, by the preceding, N is the pole of PM, and therefore &c.

Every triangle MPN thus related to a circle, that its three vertices and their opposite sides are pole and polar to each other, is said (for a reason that will presently appear) to be *self-reciprocal* with respect to the circle; and it is evident from the definition of pole and polar, in Art. 165, that *in every self-reciprocal triangle with respect to a circle, the three perpendiculars from the vertices upon the opposite sides intersect at the centre, and are there divided so that the rectangle under the segments of each = the square of the radius of the circle.*

Since in every triangle ABC the three perpendiculars AX, BY, CZ from the vertices upon the opposite sides intersect at a common point O for which the three rectangles $OA.OX$, $OB.OY$, $OC.OZ$ are equal in magnitude and sign; therefore, by the above, *every triangle ABC is self-reciprocal with respect to the circle whose centre is the intersection O of the three perpendiculars AX, BY, CZ from its vertices on its opposite sides and the square of whose radius is the common value of the three equal rectangles $OA.OX$, $OB.OY$, $OC.OZ$, and which is therefore real or imaginary according as that common value is positive or negative, that is, according as the triangle is obtuse or acute angled.*

In the particular case of a right-angled triangle of any finite magnitude, the point O being the vertex of the right angle, and the common value of the three rectangles $OA.OX$, $OB.OY$, $OC.OZ$ being $= 0$; hence, from the above, *every right-angled triangle of finite magnitude is self-reciprocal with respect to the circle of evanescent radius whose centre is the vertex of the right angle.*

If, while the vertex of the right angle remains at a finite distance, the opposite side be conceived to recede to infinity; since, then, the common value of the three rectangles $OA.OX$, $OB.OY$, $OC.OZ$ is indeterminate (13), hence, again, from the above, *every right-angled triangle whose hypotenuse is at infinity is self-reciprocal with respect to every circle of finite radius whose centre is the vertex of the right angle.*

For any triangle ABC, whatever be its magnitude and form,

if A, B, C be its three angles, and d the diameter of its circumscribing circle; the square of the radius OR of the circle to which it is self-reciprocal is given in all cases by the formula

$$OR^2 = -d^2 . \cos A . \cos B . \cos C,$$

which, as the cosine of a right angle is evanescent, includes evidently with all others the two particular cases just noticed.

For, since for its centre O, which, in virtue of the property of the present article, is termed *the polar centre*, as the circle itself is, for the same reason, *the polar circle* of the triangle, the three circles BOC, COA, AOB are all equal to the circle ABC, therefore, by (62, Cor. 7°) and by (62), disregarding signs,

$$OX = OB . OC \div d, \quad OY = OC . OA \div d, \quad OZ = OA . OB \div d,$$

and $\quad OA = d . \cos A, \quad OB = d . \cos B, \quad OC = d . \cos C,$

therefore

$$OR^2 = OX . OA = OY . OB = OZ . OC$$
$$= OA . OB . OC \div d = d^2 . \cos A . \cos B . \cos C,$$

and as the two magnitudes thus shewn to be always equal in absolute value are evidently always opposite in sign, therefore &c.

If a, b, c be the three sides of the triangle, and d, as before, the diameter of its circumscribing circle, it is easy to see from the above, or directly, that also

$$OR^2 = \tfrac{1}{2} (a^2 + b^2 + c^2) - d^2,$$

which is the formula for the square of the radius of the polar circle in terms of the three sides of the triangle.

In every triangle the polar circle, real or imaginary, intersects at right angles the three circles, of which the three sides are diameters. For the extremities of each perpendicular of the triangle being inverse points with respect to the polar circle (149), and the circle on each side as diameter passing through the four extremities of the two perpendiculars to the other two sides (Euc. III. 31), therefore &c. (156).

169. *When two triangles, whose vertices and sides correspond in pairs, are such with respect to a circle, that each vertex of one is the pole of the corresponding side of the other, or conversely,*

then, reciprocally, each vertex of the latter is the pole of the corresponding side of the former, or conversely.

For if P, Q, R be the three vertices of either triangle, and L', M', N' the three corresponding sides of the other; then, since, by hypothesis, P is the pole of L', Q of M', R of N', therefore, by (167), QR is the polar of $M'N'$, RP of $N'L'$, PQ of $L'M'$, and therefore &c.

More generally, *when two polygons of any order are such with respect to a circle, that every vertex of one is the pole of a corresponding side of the other, or conversely, then, reciprocally, every vertex of the latter is the pole of a corresponding side of the former, or conversely.*

For, if P, Q, R, S, &c. be the several vertices of either polygon, and L', M', N', O', &c. the several corresponding sides of the other; then, since, by hypothesis, P is the pole of L', Q of M', R of N', S of O', &c., therefore, by (167), PQ is the polar of $L'M'$, QR of $M'N'$, RS of $N'O'$, &c., and therefore &c.

More generally still, *when two figures of any nature are such with respect to a circle, that every point of one is the pole of a corresponding tangent to the other, or conversely, then, reciprocally, every point of the latter is the pole of a corresponding tangent to the former, or conversely.*

For if P and Q be any two points of either figure F, and L' and M' the two corresponding tangents to the other F', then, since, by hypothesis, P is the pole of L' and Q of M', therefore by (167), PQ is the polar of $L'M'$; and this being true in all cases, whatever be the separation of Q from P or of M' from L', is therefore true in the particular case when Q coincides with P, and consequently M' with L'; that is, when (19) PQ is the tangent L at the point P to the figure F, and when (20) $L'M'$ is the point of contact P' of the tangent L' with the figure F', and therefore &c.

170. Every two triangles, polygons, or figures of any kind F or F' then reciprocally related to each other, that the several points of either and the corresponding lines of the other are pole and polar to each other with respect to a circle, are said, each to be *the polar* of the other, and both together to be *reciprocal polars* to each other, with respect to the circle; the reciprocality

between them consisting in the circumstance, above established, that when either is the polar of the other with respect to a circle, then, reciprocally, the latter is the polar of the former with respect to the same circle.

Two polygons of any order, one inscribed and the other circumscribed to a circle at the same system of points on its circumference, furnish an obvious example of a pair of polygons reciprocal polars to each other with respect to the circle; the vertices and sides of the former being respectively the points of contact of the sides and the chords of contact of the angles of the latter. Two concentric circles again furnish another obvious example of a pair of figures, reciprocal polars to each other with respect to the concentric circle the square of whose radius equals the rectangle under their radii; either being indifferently the locus of the poles of all the tangents to the other, or the envelope of the polars of all the points of the other, with respect to that circle.

A figure of any nature F is said to be *self-reciprocal* with respect to a circle, when its several points and lines correspond in pairs pole and polar to each other with respect to the circle; thus, as stated in (168), every triangle ABC is self-reciprocal with respect to the particular circle, real or imaginary, to which its vertices and opposite sides are pole and polar to each other.

If either of two figures of any nature, reciprocal polars with respect to any circle, be turned round the centre of the circle into the opposite position, the two figures will then evidently be reciprocal polars with respect to the concentric circle the square of whose radius is equal in magnitude and opposite in sign to that of the original circle; of the two circles, for the two opposite positions, one therefore is always real and the other always imaginary.

171. Every two figures F and F', reciprocal polars to each other with respect to a circle, possess evidently (165 and 166) the following reciprocal properties:

1°. Every line L of either is perpendicular to that connecting the corresponding point P' of the other with the centre of the circle; and conversely.

2°. The angle determined by any two lines L and M of either

is similar to that subtended by the two corresponding points P' and Q' of the other at the centre of the circle; and conversely.

3°. When of three lines L, M, N of either, two make equal angles with the third, then of the lines connecting the three corresponding points P', Q', R' of the other with the centre of the circle, the corresponding two make equal angles with the third; and conversely.

4°. The rectangle under the distances of any point P of either and of the corresponding line L' of the other from the centre of the circle is constant; and conversely.

5°. When two points P and Q of either are equidistant from the centre of the circle, the two corresponding lines L' and M' of the other are equidistant from the centre of the circle; and conversely.

6°. When three points P, Q, R of either are collinear, the three corresponding lines L', M', N' of the other are concurrent; and conversely.

172. Any figure F being given or taken arbitrarily, its polar F' with respect to any circle can always be derived from it, by the simple construction of taking either the polars of its several points or the poles of its several lines with respect to the circle; and the repetition of the same process to the new figure F', thus determined by either construction, always (169) reproduces the original figure F; thus, every figure F, whatever be its nature, has its polar figure F' with respect to every circle, and every two figures F and F', reciprocal polars to each other with respect to any circle, always produce and reproduce each other alternately by continued repetition of either process by which one may be derived from the other.

The process of transformation, just described, by which all the points of a figure of any nature are changed into their polars, and all the lines of the figure into their poles, with respect to an arbitrary circle, is sometimes termed *polarization*, the circle by aid of which it is performed *the polarizing circle*, and the centre and radius of the circle *the centre and radius of polarization;* but from the reciprocality, as above explained, existing between the original and derived figures, the process of transformation is more generally known as *reciprocation*, the

circle by aid of which it is performed as *the reciprocating circle*, and the centre and radius of the circle as *the centre and radius of reciprocation.*

In the process of reciprocation, the reciprocating circle, provided only it be of a finite radius and at a finite distance, being otherwise entirely arbitrary as to magnitude and position, should of course, when necessary, be selected so as to accord most conveniently with the circumstances of the case; as, for instance, if it were required to obtain the reciprocal of any property of a single circle as far only as another property of a single circle is concerned, the circle itself, or at least one concentric with it, should be made the reciprocating circle, as one not concentric with it would transform it by reciprocation into a figure of more general form than a circle; or, if it were convenient for any reason to have any point or line of the reciprocal figure at infinity, the centre of the reciprocating circle should be placed on the corresponding line or at the corresponding point of the original figure, as any other position of its centre would leave the point or line in question at a finite distance (165, 3°, 5°); thus, a tetrastigm in its general form reciprocates into a tetragram in its general form, into a trapezium, or into a parallelogram, according as the centre of reciprocation is arbitrary, on any one of its six lines of connection, or at the vertex of any one of its three angles of connection (107); a circle, as above stated, reciprocates into a figure of more general form or into a circle, according as the centre of reciprocation is arbitrary or at its centre; and similarly, for figures of all kinds, the reciprocals of whose properties adapted to reciprocation are often much simplified by a convenient selection of the reciprocating circle.

173. As figures consisting of combinations of points and lines give by reciprocation to every circle figures consisting of combinations of lines and points, *all properties of such figures adapted to reciprocation are accordingly double, and from either of two reciprocal properties established for such a figure the other may always be inferred without further demonstration;* thus, from the Theorem of Pascal (148, *a*), that "in every hexagon inscribed in a circle the three pairs of opposite sides intersect

collinearly," may be, and in fact originally was, derived, by reciprocation to the circle, the Theorem of Brianchon (148, *b*), that "in every hexagon circumscribed to a circle the three pairs of opposite vertices connect concurrently," or conversely, (see 171, 6°)—*Hence one very important use of the reciprocating process as enabling us at once to double our previous knowledge of all properties adapted to reciprocation in the geometry of the point and line.*

Again, as circles give by reciprocation to circles not concentric with themselves figures of more general forms than circles, *all properties of circles obtained by reciprocation are consequently true of the more general figures derived from them by reciprocation, and from either of two reciprocal properties established for a circle, the other may always be inferred without further demonstration for the more general figures into which the circle reciprocates for different positions of the centre of reciprocation;* thus, from either of the two aforesaid reciprocal properties of Pascal and Brianchon established for the circle, the other may be inferred without further demonstration for every variety of figure into which the circle reciprocates—*Hence another and still more important use of the reciprocating process, as enabling us to evolve from the familiar and comparatively simple properties of the circle adapted to reciprocation, all the reciprocal properties for the more general figures into which the circle becomes transformed by reciprocation.*

In a treatise confined like the present to the geometry of the point, line, and circle, any examples of the reciprocating process in its second and higher use cannot of course be given, nor would they be intelligible to the reader without some previous knowledge of the Theory of Conic Sections; in its other use, however, examples of reciprocal properties of elementary figures, grouped in reciprocal pairs, marked by corresponding numbers or letters, but independently established, will be found in considerable numbers all through the advanced chapters of the work; the process of connecting the several pairs by reciprocation as they occur, thus furnishing a continued and very valuable exercise to the reader.

174. The two fundamental properties of Art. 167, from which the important consequences of the several succeeding

Articles have been inferred, may obviously be stated otherwise thus, as follows—

When two points are such that one lies on the polar of the other with respect to a circle, then, reciprocally, the latter lies on the polar of the former with respect to the circle; and, conversely, when two lines are such that one passes through the pole of the other with respect to a circle, then, reciprocally, the latter passes through the pole of the former with respect to the circle.

For, as there proved, see figures of that Article, when M lies on L then P lies on XY, and, conversely, when XY passes through P then L passes through M, and therefore &c.

Every two points thus related to each other, that each lies on the polar of the other with respect to a circle, are termed *conjugate points* with respect to the circle; and every two lines thus related to each other, that each passes through the pole of the other with respect to a circle, are termed *conjugate lines* with respect to the circle; in the figures of Art. 166 the two points M and N are evidently conjugate points, and the two lines PM and PN are evidently conjugate lines with respect to the circles.

From 5°, Art. 165, it is evident that—*Every two points at infinity in directions at right angles to each other are conjugate points with respect to every circle, and every two lines at right angles to each other are conjugate lines with respect to every circle whose centre is the intersection of the lines.*

175. Conjugate points and lines with respect to a circle possess evidently, see figures of Art. 166, the following general properties—

1°. Every point has an infinite number of conjugates, viz. all points lying on its polar; and, every line has an infinite number of conjugates, viz. all lines passing through its pole.

2°. When two points are conjugate so are their polars; and, conversely, when two lines are conjugate so are their poles.

3°. The common conjugate to any two points is the pole of their line of connection; and, conversely, the common conjugate to any two lines is the polar of their points of intersection.

4°. The lines by which two conjugate points connect with the pole of their line of connection are the polars of the points;

and, conversely, the points at which two conjugate lines intersect with the polar of their point of intersection are the poles of the lines.

5°. Every two conjugate points connect with the pole of their line of connection by a pair of conjugate lines; and, conversely, every two conjugate lines intersect with the polar of their point of intersection at a pair of conjugate points.

6°. Every two conjugate points determine with the pole of their line of connection a self-reciprocal triangle (168); and, conversely, every two conjugate lines determine with the polar of their point of intersection a self-reciprocal triangle (168). Hence, every self-reciprocal triangle with respect to a circle is said also to be *self-conjugate* with respect to the circle.

176. For every pair of conjugate points with respect to a circle the following metric relations exist, each of which reciprocally determines a pair of conjugate points with respect to the circle.

1°. *The square of the distance between them is equal to the sum of the squares of the tangents from them to the circle.*

2°. *The semi-distance between them is equal to the length of the tangent from its middle point to the circle.*

3°. *The rectangle under their distances from the polar centre of their line of connection is equal in magnitude and opposite in sign to the square of the tangent from that point to the circle.*

For, if M and N (figures, Art. 166) be any two points, O and Q the middle point and polar centre of their line of connection, C the centre of the circle, and P the intersection of the three perpendiculars MF, NE, and CQ of the triangle MCN, then—

To prove 1° and its converse. Since, by Euc. II. 12, 13, $MN^2 = CM^2 + CN^2 - 2\,CM.CE$ or $-2\,CN.CF$; when M and N are conjugate points, and when therefore $CM.CE$, or its equivalent $CN.CF$, = rad^2 of circle, then $MN^2 = CM^2 + CN^2 - 2$ rad^2 of circle $= (CM^2 - \text{rad}^2) + (CN^2 - \text{rad}^2) = \tan^2$ from $M + \tan^2$ from N; and, conversely, when the latter relation holds, then $CM.CE$, or its equivalent $CN.CF$, = rad^2 of circle, and therefore M and N are conjugate points with respect to the circle.

To prove 2° and its converse. Since, by 98, or Euc. II. 12, 13, Cor.,
$$CM^2 + CN^2 = OM^2 + ON^2 + 2\,OC^2,$$

and consequently

$$CM^2 + CN^2 - 2 \text{ rad}^2 \text{ of circle} = OM^2 + ON^2 + 2 \tan^2$$

from O to circle; when M and N are conjugate points, and when therefore, by 1°,

$$CM^2 + CN^2 - 2 \text{ rad}^2 \text{ of circle} = MN^2 = 2\,(OM^2 + ON^2),$$

then $OM^2 + ON^2 = 2 \tan^2$ from O to circle, and therefore $OM^2 = ON^2 = \tan^2$ from O to circle; and, conversely, when the latter relation exists, then $CM^2 + CN^2 - 2 \text{ rad}^2$ of circle $= MN^2$, and therefore, by 1°, M and N are conjugate points with respect to the circle.

To prove 3° and its converse. Since, by either pair of similar right-angled triangles MQP and CQN, or NQP and CQM, the two ratios $QM : QP$ and $QC : QN$, and therefore the two rectangles $QM.QN$ and $QP.QC$, are equal in magnitude and opposite in sign; when M and N are conjugate points, and when therefore (174) P is the pole of MN, then the latter rectangle (165) is equal in magnitude and sign to the square of the tangent from Q to the circle; and, conversely, when the latter rectangle is equal in magnitude and sign to the square of that tangent, then (165) P is the pole of MN, and therefore (174) M and N are conjugate points with respect to the circle.

In the particular case when the radius of the circle is evanescent, the above properties all follow immediately from the obvious consideration (168) that every two conjugate points with respect to an evanescent circle subtend a right angle at the centre of the circle, and that, conversely, every two points which subtend a right angle at the centre of an evanescent circle are conjugate points with respect to the circle.

177. *Every circle having for diameter the interval between two conjugate points with respect to another circle is orthogonal to the other.*

For, the circle on MN as diameter (figures, Art. 166) passes evidently through the two points E and F, which are the inverses of M and N when the latter are conjugates with respect to the circle C, and therefore &c. (156).

Conversely, *When two circles intersect at right angles, the extremities of every diameter of either are conjugate points with respect to the other.*

For, MN (same figures) being any diameter of either, C the centre of the other, and E and F the two points in which the former intersects the two diameters CM and CN of the latter; since then (156) E and F are the two inverses of M and N with respect to the latter, therefore (165) EN and FM are the two polars of M and N with respect to the same, and therefore &c. (174).

COR. 1°. The above property is evidently identical with 2° of the preceding Article, and from either it obviously follows immediately that—

1°. *The line connecting any two conjugate points with respect to a circle may be turned round its middle point through any angle without its extremities ceasing to be conjugate points with respect to the circle.*

2°. *When the distance between two conjugate points with respect to a circle of given radius is given, the distance of their middle point from the centre of the circle is also given, and conversely.*

3°. *If the same circle be orthogonal to a number of others, the extremities of every diameter of it are conjugate points with respect to all the others.*

4°. *The locus of points having a common conjugate with respect to three circles is the circle intersecting the three at right angles.*

COR. 2°. Since, when two points are conjugates with respect to a number of circles, the polars of either with respect to them all pass through the other (174); hence, from 3° and 4°, Cor. 1°—

1°. *If the same circle be orthogonal to a number of others, the polars of every point on it with respect to them all pass through the diametrically opposite point.*

2°. *The locus of points whose polars with respect to three circles are concurrent is the circle intersecting the three at right angles.*

COR. 3°. By aid of 156, Cor. 3°, the above supply obvious solutions of the four following problems—

1°. *On a given line or circle to determine two points separated by a given interval which shall be conjugates with respect to a given circle.*

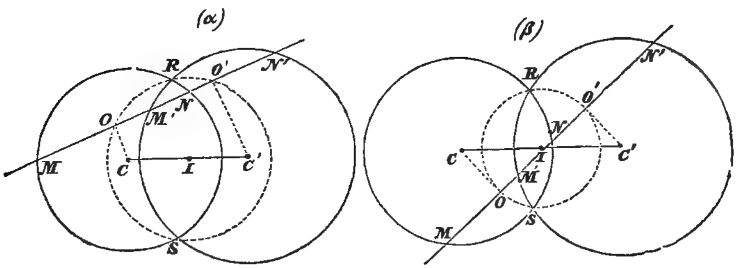
2°. *On a given line or circle to determine two points which shall be at once conjugates with respect to two given circles.*

178. *When a line intersecting two circles meets either in a pair of conjugate points with respect to the other.*

1°. *Then reciprocally it meets the latter in a pair of conjugate points with respect to the former.*

2°. *Its two segments intercepted by them are bisected by the circle passing through their points of intersection whose centre bisects the distance between their centres.*

3°. *The rectangle under its distances from their centres is equal in magnitude and sign to half the sum of the squares of their radii — half the square of the distance between their centres.*



For, if C and C' be the centres of the two circles, R and S their two points of intersection, MN and $M'N'$ the two segments they intercept on the line, O and O' the two middle points of the segments, and I the middle point of CC'; then—

To prove 1°. The relation $OM'.ON' = (\frac{1}{2}MN)^2$, or the equivalent relation $O'M.O'N = (\frac{1}{2}M'N')^2$, (Euc. II. 5, 6), being at once the condition (176, 2° and 3°) that M and N should be conjugate points with respect to the circle C', and that M' and N' should be conjugate points with respect to the circle C, therefore &c.

To prove 2°. Since

$$OM'.ON' = (\tfrac{1}{2}MN)^2 \text{ and } O'M.O'N = (\tfrac{1}{2}M'N')^2,$$

therefore $$C'O^2 - C'R^2 = CR^2 - CO^2,$$

and $$CO'^2 - CR^2 = C'R^2 - C'O'^2;$$

therefore $$CO^2 + C'O^2 = CO'^2 + C'O'^2 = CR^2 + C'R^2,$$

from which it follows, by (98, Cor. 2°.), that O, O', and R lie on the same circle having I for centre, and therefore &c.

To prove 3°. Since OC and $O'C'$ are perpendiculars at the

extremities of the chord OO' of the circle ORO', meeting the diameter CC' at the points C and C' equidistant from the centre I; therefore (49)

$$CO.C'O' = IR^2 - (\tfrac{1}{2}CC')^2 = \tfrac{1}{2}(CR^2 + C'R^2 - CC'^2), (83, \text{Cor. } 2°.);$$

and therefore &c.

Cor. 1°. In the particular case when the two circles intersect at right angles, since then (23) $CR^2 + C'R^2 = CC'^2$, therefore, from the above (3°), $CO.C'O'=0$; and therefore, as proved otherwise in the preceding Article—

When two circles intersect at right angles every line intersecting either in a pair of conjugate points with respect to the other passes through one of their centres.

Cor. 2°. The above (2° and 3°) supply obvious solutions of the two following problems—

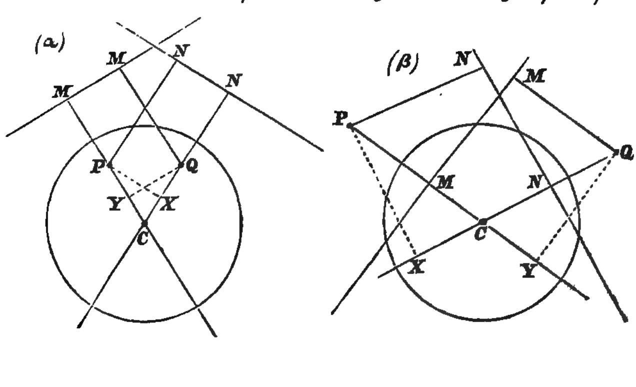
1°. *Through a given point to draw a line intersecting one of two given circles in a pair of conjugate points with respect to the other.*

2°. *To draw a line intersecting two of three given circles in pairs of conjugate points with respect to the third.*

179. In connection with the subject of poles and polars with respect to the circle, the following useful theorem is due to Dr. Salmon.

The distances of any two points from the centre of a circle have the same ratio as their distances each from the polar of the other with respect to the circle.

If P and Q be the two points, M and N their two polars, and C the centre of the circle, then $PC : QC = PN : QM$; for, letting



fall from P and Q the perpendiculars PX and QY upon the diameters CQ and CP, then since (165)

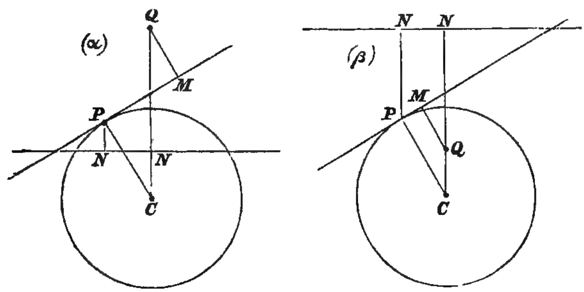
$$CP.CM = CQ.CN = \text{rad}^2 \text{ of circle,}$$

therefore $CP : CQ = CN : CM = CX + PN : CY + QM$,

but, by similar right-angled triangles, $CP : CQ = CX : CY$, therefore $CP : CQ = PN : QM$, and therefore &c.

In the particular case when $CP = CQ$, it is evident without proof that then $PN = QM$, or, in general, that—*Every two points equidistant from the centre of a circle are equidistant each from the polar of the other;* and, in particular, that—*Every two points on the circumference of a circle are equidistant each from the tangent at the other.*

Cor. 1°. If one of the two points Q, in the above, with its polar N, be supposed fixed and arbitrary, and the other P, with its polar M, variable and confined to the circumference of the circle; since then the ratio $CP : CQ$ is constant, therefore,



by the above, its equivalent $PN : QM$ is also constant, and therefore, the polar of any point on a circle being the tangent at the point,

The distance of a variable point on a fixed circle from any fixed line is to the distance of the tangent at the point from the pole of the line in the constant ratio of the radius of the circle to the distance of the pole from its centre.

Cor. 2°. The following among many consequences follow immediately from Cor. 1°—

1°. The product of any number of constant ratios being of course constant, therefore—

The rectangle under the distances of a variable point on a

fixed circle from any two fixed lines is to the rectangle under the distances of the tangent at the point from the poles of the lines in the constant ratio of the square of the radius of the circle to the rectangle under the distances of the poles from its centre.

2°. Every two polygons reciprocal polars to each other with respect to a circle (170) being such that the vertices of either and the corresponding sides of the other are pole and polar to each other with respect to the circle, therefore—

For every two polygons, reciprocal polars to each other with respect to a circle, the product of the distances of any point on the circle from the n sides of either is to the product of the distances of the tangent at the point from the n vertices of the other in the constant ratio of the n^{th} *power of the radius of the circle to the product of the distances of the n vertices from its centre.*

3°. For every two polygons, one inscribed and the other circumscribed to a circle at the same system of points on its circumference (polygons which evidently come under the preceding head) the products of the distances of the two sets of sides from any point on the circle being equal (48, Ex. 9°.), therefore—

For every two polygons, one inscribed and the other circumscribed to a circle at the same system of points on its circumference, the products of the distances of the two sets of vertices from any tangent to the circle have the constant ratio of the products of their distances from the centre of the circle.

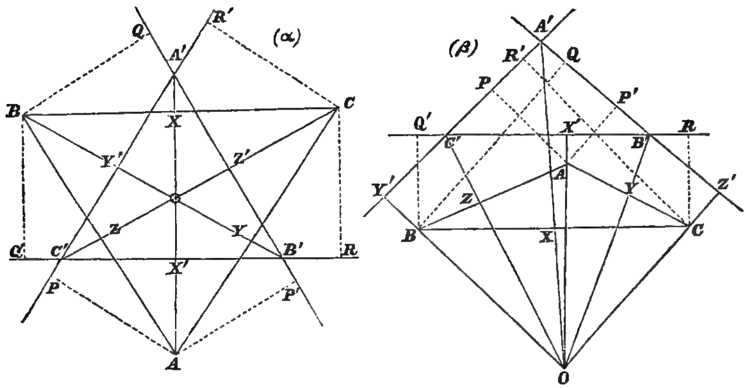
4°. In every tetrastigm inscribed in a circle, the rectangles under the distances of the three pairs of opposite connectors from any point on the circle being equal (62, Cor. 10°.), therefore—

In every tetragram circumscribed to a circle, the rectangles under the distances of the three pairs of opposite intersections from any tangent to the circle have the constant ratios of the rectangles under their distances from the centre of the circle.

180. With the three following properties of two triangles, reciprocal polars to each other with respect to a circle, we shall close the present chapter.

1°. *Every two triangles reciprocal polars to each other with respect to a circle are in perspective* (140).

For, if ABC and $A'B'C'$ be the two triangles, and O the



centre of the circle, from the three vertices A, B, C of either triangle, letting fall the three pairs of perpendiculars AP and AP', BQ and BQ', CR and CR' upon the three pairs of sides about the corresponding vertices of the other $A'B'C'$; then, since by Dr. Salmon's Theorem (179),

$$\frac{BQ}{CR'} = \frac{OB}{OC}, \quad \frac{CR}{AP'} = \frac{OC}{OA}, \quad \frac{AP}{BQ'} = \frac{OA}{OB},$$

therefore, at once, by composition

$$\frac{AP}{AP'} \cdot \frac{BQ}{BQ'} \cdot \frac{CR}{CR'} = 1,$$

and therefore &c. (140).

In the particular case of two triangles, one inscribed and the other circumscribed to a circle at the same three points on its circumference, this general property obviously gives at once the two reciprocal properties established on other principles in Examples 3° and 4°, Art. 137. See also (148), where the same properties have been already inferred as particular cases from the general theorems of Pascal and Brianchon, respecting any hexagons inscribed and circumscribed to a circle.

2°. *Any two triangles, however circumstanced as to magnitude and form, may be placed relatively to each other, so as for any assigned correspondence of vertices and sides to be reciprocal polars with respect to a circle; and that in one or in three pairs of opposite positions* (170), *according as the two sets of corresponding vertices are disposed in similar or opposite directions of rotation round the two triangles.*

For, that the two triangles ABC and $A'B'C'$ (same figures as before) should be reciprocal polars with respect to a circle, real or imaginary, it is sufficient that the three perpendiculars AX', BY', CZ' from the vertices of ABC upon the corresponding sides of $A'B'C'$ pass through a common point O, and that the three $A'X$, $B'Y$, $C'Z$ from the vertices of $A'B'C'$ upon the corresponding sides of ABC pass through the same point O (133, Ex. 7°.); those conditions securing (by pairs of similar triangles, see figures), that the six rectangles $OA.OX'$, $OB.OY'$, $OC.OZ'$, $OA'.OX$, $OB'.OY$, $OC'.OZ$ shall be equal in magnitude and sign; taking therefore, according as the corresponding vertices of the two triangles are disposed in similar or opposite directions of rotation, as in figs. α and β respectively, for the triangle ABC, the internal or one of the three external points O for which the three angles BOC, COA, AOB are similar to the three $B'A'C'$, $C'B'A'$, $A'C'B'$, and for the triangle $A'B'C'$, the internal or corresponding external point O' for which the three angles $B'O'C'$, $C'O'A'$, $A'O'B'$ are similar to the three BAC, CBA, ACB (63, Cor. 4°.); and then placing the two triangles, so that the two points O and O' shall coincide, and that the six connectors OA, OB, OC, OA', OB', OC' shall be similar or opposite in direction with the six perpendiculars OX', OY', OZ', OX, OY, OZ, the required position is obtained; the circle, to which the triangles are polars, being real in the former case and imaginary in the latter (170).

In the particular case when the two triangles are similar, and when the correspondence is between their homologous vertices and sides, the two points O and O', evidently homologous points with respect to the triangles (39), are, for similar directions of rotation, fig. α, the two points of concurrence of their two sets of perpendiculars (63, Cor. 5°.), and for opposite directions of rotation, fig. β, any two homologous points on their circumscribing circles (63, Cor. 5°.); hence, as is also evident directly—*Every triangle reciprocates into a similar triangle to every circle whose centre is either the unique point of concurrence of its three perpendiculars or any point indifferently on its circumscribing circle; the two similar triangles being both right or left in the former case, and one right and one left in the latter* (32); in the former case also their homologous sides being evidently parallel, they are

consequently similarly or oppositely placed (33), thus verifying for their particular case the general property 1°, see (142).

3°. *If ABC be any triangle, A'B'C' its polar triangle with respect to any circle, O the centre and OR the radius of the circle, then*

$$(A'B'C') = \frac{OR^4}{4} \cdot \frac{(ABC)^2}{(BOC).(COA).(AOB)},$$

and similarly for (ABC) in terms of (A'B'C'); the quantities within the parentheses signifying the areas of the several triangles they respectively represent.

For, since, by pairs of similar angles (64), (same figures as before)

$$\frac{(B'OC')}{(ABC)} = \frac{OB'.OC'}{AB.AC}, \quad \frac{(C'OA')}{(ABC)} = \frac{OC'.OA'}{BC.BA}, \quad \frac{(A'OB')}{(ABC)} = \frac{OA'.OB'}{CA.CB},$$

and, since, by (165),

$$OA'.OX = OB'.OY = OC'.OZ = OR^2,$$

therefore

$$(B'OC') = \frac{OR^4.(ABC)}{(AB.OZ).(AC.OY)} = \frac{OR^4}{4} \cdot \frac{(ABC)}{(AOB).(AOC)},$$

$$(C'OA') = \frac{OR^4.(ABC)}{(BC.OX).(BA.OZ)} = \frac{OR^4}{4} \cdot \frac{(ABC)}{(BOC).(BOA)},$$

$$(A'OB') = \frac{OR^4.(ABC)}{(CA.OY).(CB.OX)} = \frac{OR^4}{4} \cdot \frac{(ABC)}{(COA).(COB)};$$

and therefore, by addition, remembering whatever be the position of O (118), that

$$(B'OC') + (C'OA') + (A'OB') = (A'B'C'),$$

and that $$(BOC) + (COA) + (AOB) = (ABC),$$

the above relation is the evident result.

It is evident from the above, that for a given triangle ABC, and for a circle of given radius, but variable centre O, the area of the polar triangle $A'B'C'$ varies inversely as the product of the three areas (BOC), (COA), (AOB), and is therefore a minimum when that product is a maximum, that is (57, Ex. 3°.), when its three factors, their sum being constant (118), are equal, or when (91, Cor.) O is the mean centre of the three points A, B, C for multiples all $= 1$.

It may also be readily shewn from the same that—*In every triangle the polars of the middle points of the sides with respect to the inscribed circle determine a triangle equal in area to the original;* for a, b, c being the three sides, α, β, γ the three perpendiculars, s the semi-perimeter, and r the radius of the inscribed circle of any triangle; if A, B, C be the middle points of its sides, and O the centre of its inscribed circle, the three areas (BOC), (COA), (AOB), in the above, are easily seen, on drawing a figure, to be equal to the three products

$$\left(\frac{\alpha}{2}-r\right)\frac{a}{4}, \quad \left(\frac{\beta}{2}-r\right)\frac{b}{4}, \quad \left(\frac{\gamma}{2}-r\right)\frac{c}{4},$$

from which, since

$$a\alpha = b\beta = c\gamma = 2sr = 2 \text{ area of } abc = 8\,(ABC),$$

it follows, without difficulty, from the above, that

$$(A'B'C') = \frac{s^3r^3}{s\,(s-a)\,(s-b)\,(s-c)} = \frac{\text{area}^3 \text{ of } abc}{\text{area}^2 \text{ of } abc} = \text{area of } abc,$$

and therefore &c.

CHAPTER XI.

ON THE RADICAL AXES OF CIRCLES CONSIDERED IN PAIRS.

181. THE line intersecting at right angles the common diameter of two circles, and dividing the interval AB between their centres A and B at the point I for which the difference of the squares of the segments $AI^2 - BI^2$ is equal in magnitude and sign to the difference of the squares of the conterminous radii $AR^2 - BS^2$, is termed *the radical axis* of the circles.

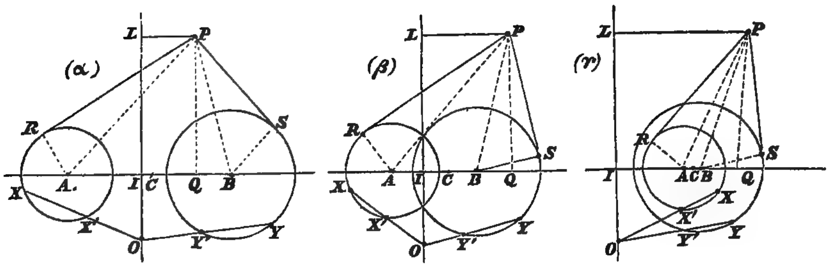
From the mere definition of the radical axis of two circles, it is evident that: 1°. when the circles intersect, it passes through the two points of intersection (Euc. I. 47); 2°. when they touch, it touches both at the point of contact (Euc. III. 16); 3°. when they are equal and not concentric, it coincides with their axis of reflexion (50); 4°. when they are concentric and not equal, it coincides with the line at infinity (136, 1°); 5°. when they are at once equal and concentric, it is indeterminate (13); 6°. when one is a line and the other not, it coincides with the line (150); 7°. when one is a point and the other not, it coincides with the axis of reflexion of the point and its inverse with respect to the other (157); and 8°. when they are both points or lines, the case comes under the head of 3°. or of 5°. Of these particulars, some, less evident than the others, will appear more fully from the general properties of the radical axis of any two circles, which will form the main subject of the present chapter.

When two circles, whatever be their nature, are given in magnitude and position, their radical axis, when not indeterminate, is of course implicitly given with them; the relation $AI^2 - BI^2 = AR^2 - BS^2$ fixing, evidently, the position, when determinate, of the point I at which it intersects at right angles their line of centres AB.

182. Of all the properties of the radical axis of two circles, the following leads to the greatest number of consequences, and may be regarded as fundamental.

The difference of the squares of the tangents from any point to two circles = twice the rectangle under the distance between their centres and the distance of the point from their radical axis.

For, if A and B be the centres of the two circles, AR and



BS their radii, IL their radical axis, P the point, PR and PS the tangents from it to the circles, PL and PQ the perpendiculars from it on IL and AB, and C the middle point of AB; then, since, (Euc. I. 47),

$$PR^2 = AP^2 - AR^2 \text{ and } PS^2 = BP^2 - BS^2,$$

therefore

$$(PR^2 - PS^2) = (AP^2 - BP^2) - (AR^2 - BS^2), \text{ but, (Euc. I. 47),}$$

$$AP^2 - BP^2 = AQ^2 - BQ^2 = 2AB.CQ, \text{ (Euc. II. 5, 6),}$$

and, by the definition of the radical axis (181),

$$AR^2 - BS^2 = AI^2 - BI^2 = 2AB.CI, \text{ (Euc. II. 5, 6),}$$

therefore

$$(PR^2 - PS^2) = 2AB.(CQ - CI) = 2AB.IQ = 2AB.LP,$$

and therefore &c.

Cor. 1°. If $PL = 0$, then $PR^2 - PS^2 = 0$; and, conversely, if $PR^2 - PS^2 = 0$, then $PL = 0$. Hence—*Tangents to two circles from any point on their radical axis are equal; and, conversely, when tangents to two circles from a point not at infinity are equal, the point is on the radical axis of the circles.*

It is this property, of which that of (157) is evidently a particular case, which has given the name "Radical axis" to the

line in question, the tangents to two circles from any point being expressed by *radicals*, and the locus of points for which they are equal being a *line*.

The two tangents to the same circle from any point being equal, it follows of course from the second part of the above, that—

The tangents to two circles at their points of contact with any circle touching both intersect on their radical axis.

Cor. 2°. If $PS = 0$, then $PR^2 = 2AB.LP$, and conversely, if $PR^2 = 2AB.LP$, then $PS = 0$. Hence—

The square of the tangent to either of two circles from any point on the other varies as the distance of the point from their radical axis; and, conversely, when the square of the tangent from a point to a circle varies as the distance of the point from a line, the point lies on another circle, of which and the original the line is the radical axis.

Of this property, that of (159) is evidently a particular case.

Cor. 3°. If O be the intersection of any two chords XX' and YY' of the circles. Since, when their four extremities are concyclic, then $OX.OX' = OY.OY'$, and conversely, (Euc. III. 35, 36); and since, by Cor. 1°., the same is the condition that the point O should be on the radical axis of the circles, and conversely. Hence—

Every two chords of two circles whose four extremities are concyclic intersect on their radical axis; and, conversely, when two chords of two circles intersect on their radical axis, their four extremities are concyclic.

This property will be stated more generally in the next article.

Cor. 4°. The point O, as before, being on the radical axis, if $OX = OY$, that is, if X and Y be two of the four intersections with the two circles of any third circle having its centre on their radical axis; then, since, by Cor. 3°., $XX' = YY'$, and since, by (62), $XX' = 2AX.\cos AXO$, and $YY' = 2BY.\cos BYO$; therefore $\cos AXO : \cos BYO = BY : AX$, and both angles having evidently the same affection (11). Hence—

Every circle having its centre on the radical axis of two others intersects them at angles, of the same affection, whose cosines are inversely as their radii; and, conversely, every circle intersecting

two others at angles, of the same affection, whose cosines are inversely as their radii, has its centre on their radical axis.

The general property, of which this is a particular case, will be given further on.

COR. 5°. In the same case, since when $XX' = 0$, then $YY' = 0$, and conversely, therefore as a particular case of Cor. 4°., or as is also evident directly from Cor. 1°.—

Every circle having its centre on the radical axis of two others, and intersecting either at right angles, intersects the other at right angles; and, conversely, every circle intersecting two others at right angles has its centre on their radical axis.

This last is the more general proof of the latter property to which allusion was made in Art. 156, Cor. 3°. of chap. IX.

COR. 6°. Whatever be the position of the point O, whether on the radical axis or not, since, by the fundamental property above,

$$OX.OX' - OY.OY' = 2AB.LO,$$

where LO is the distance of O from the radical axis, if $OX = OY$, that is, if X and Y be two of the four intersections with the two circles of any third circle having its centre at O, then

$$OX.(XX' - YY') = 2AB.OL,$$

and therefore

$$OX : OL = 2AB : XX' - YY' = AB : AX.\cos AXO - BY.\cos BYO,$$

a ratio which is constant when the two angles of intersection AXO and BYO, whatever be their affections, are constant. Hence—

If a variable circle intersect two fixed circles at two constant angles, its radius is to the distance of its centre from their radical axis in a constant ratio; and, conversely, if a variable circle, whose radius is to the distance of its centre from the radical axis of two fixed circles in a constant ratio, intersect either circle at a constant angle, it intersects the other also at a constant angle.

COR. 7°. As either angle of intersection may = 0, or = two right angles. Hence, by (23), as a particular case of the preceding—

If a variable circle touch two fixed circles, the nature of its contact with each being invariable, its radius is to the distance of its centre from their radical axis in a constant ratio; and, con-

versely, if a variable circle, whose radius is to the distance of its centre from the radical axis of two fixed circles in a constant ratio, touch in every position either circle with contact of the same species, it intersects the other at a constant angle, which may $= 0$ *or two right angles.*

Cor. 8°. The ratio $OL : OX$ being (22) the cosine of the angle, real or imaginary, at which the variable circle in Cors. 6°. and 7°. intersects the radical axis of the two fixed circles. Hence, in general, from Cor. 6°—

A variable circle intersecting two fixed circles at constant angles intersects their radical axis at a constant angle; and, conversely, a variable circle intersecting either of two fixed circles and their radical axis at constant angles intersects the other at a constant angle.

And, in particular, from Cor. 7°—

A variable circle touching two fixed circles, the nature of the contact with each being invariable, intersects their radical axis at a constant angle; and, conversely, a variable circle intersecting the radical axis of two fixed circles at a constant angle, and touching either circle with contact of invariable species, intersects the other at a constant angle, which may $= 0$ *or two right angles.*

The general property established in this corollary is but a particular case of another still more general, which will be given in a subsequent article of the present chapter.

Cor. 9°. It is immediately evident from Cor. 1°. that—

The radical axis of two circles bisects the four segments of their four common tangents, real or imaginary, intercepted between their points of contact with the circles; and, conversely, the line joining the middle points of the intercepted segments of any two of the four common tangents to two circles, or, more generally, any two points the tangents from which to two circles are equal, is the radical axis of the circles.

And, from the first part of this latter property, that—

The two chords of contact of two circles with each pair of their common tangents, external and internal, are equidistant in opposite directions from their radical axis; and so, therefore, are the two chords for both pairs in the two circles from each other.

Cor. 10°. Since when two circles intersect at right angles,

their chord of intersection is the polar of the centre of each with respect to the other (165, 6°.). Hence from Cors. 5°. and 3°. see (166)—

The chords of intersection with two circles of every circle orthogonal to both pass through the poles of their radical axis.

The polars with respect to two circles of any point on their radical axis intersect on their radical axis.

This latter property is evidently true also of the line at infinity, a line which we shall see, in the sequel, possesses with respect to two circles nearly all the properties of their radical axis.

183. The following general property of any three circles includes evidently the first part of that established in Cor. 3°. of the preceding article as a particular case, viz.—

The three radical axes of any three circles, taken two and two, intersect at a common point, termed the radical centre of the circles.

For, if A, B, C be the three centres of the circles, AR, BS, CT their three radii, L, M, N the three radical axes of their three groups of two, and X, Y, Z, the three points in which L, M, N intersect at right angles the three sides BC, CA, AB of the triangle ABC; then since, by definition (181),

$$(BX^2 - CX^2) = (BS^2 - CT^2),$$
$$(CY^2 - AY^2) = (CT^2 - AR^2),$$
$$(AZ^2 - BZ^2) = (AR^2 - BS^2),$$

therefore

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2) = 0,$$

and therefore (132) the three perpendiculars L, M, N intersect at a common point O.

Cor. 1°. It is evident from Cors. 1°. 4°. and 5°. of the preceding, that—

1°. *The six tangents, real or imaginary, to three circles from their radical centre are equal; and, conversely, when the six tangents, real or imaginary, to three circles from a point, not at infinity, are equal, the point is their radical centre.*

2°. *Every circle having its centre at the radical centre of three others intersects them at angles, of the same affection, whose cosines are inversely as their radii; and, conversely, every circle intersecting three others at angles, of the same affection, whose cosines*

are inversely as their radii, has its centre at their radical centre.

3°. *The circle having its centre at the radical centre of three others, and intersecting one of them at right angles, intersects the other two at right angles; and, conversely, the circle intersecting three others at right angles has its centre at their radical centre.*

The obvious solution of the problem "to describe the circle which intersects three given circles at right angles," furnished by this latter property, is that to which allusion was made in Art. 156, Cor. 4°, of Chap. IX.

Since for every three chords XX', YY', ZZ' of any three circles A, B, C, which concur to their radical centre O, the three products $OX.OX'$, $OY.OY'$, $OZ.OZ'$ are always equal in magnitude and sign; their common value is sometimes termed *the radical product* of the three circles, and is, of course, always equal in magnitude and sign to the square of the radius of their orthogonal circle, which circle, consequently, is real or imaginary according as the sign of the radical product is positive or negative.

COR. 2°. *The point of concurrence O of the three perpendiculars AP, BQ, CR of any triangle ABC is the radical centre of the three circles of which the three sides BC, CA, AB are diameters.*

For, as the three circles on BC, CA, AB as diameters pass respectively through the three pairs of points Q and R, R and P, P and Q, (Euc. III. 31), therefore AP, BQ, CR are the three common chords of those circles, taken two and two, and therefore &c. (181, 1°.).

COR. 3°. *More generally, the point of concurrence O of the three perpendiculars AP, BQ, CR of any triangle ABC is the radical centre of the three circles, of which any three lines AX, BY, CZ drawn from the vertices to the opposite sides BC, CA, AB are diameters.*

For, whatever be the positions of the three diameters AX, BY, CZ, the three perpendiculars AP, BQ, CR are three chords of the three circles concurring to a point O for which the three products $OA.OP$, $OB.OQ$, $OC.OR$ are equal in magnitude and sign (168), and therefore &c. (Cor. 1°, 1°.)

COR. 4°. *For any system of three combined with any system of two circles, both systems being arbitrary.*

a. The six radical axes of the six combinations of one of the three with one of the two determine two triangles in perspective (140).

b. The radical centre of the three and the radical axis of the two are the centre and axis of the perspective (141).

For, if A, B, C be the system of three, E and F the system of two, U, V, W and X, Y, Z the two sets of three radical axes of A, B, C combined each with E and F respectively, L, M, N the three radical axes of B and C, C and A, A and B respectively, which by the above intersect at the radical centre O of A, B, C, and I the radical axis of E and F; then, by the above, the three points UX, VY, WZ lie on I, and the three pairs of points VW and YZ, WU and ZX, UV and XY lie on L, M, N respectively, and therefore &c.

The radical axis of two circles which intersect being their chord of intersection (181, 1°.), the properties just proved are consequently true, in particular, of the two triangles determined by the six chords of intersection of any two with any three circles with which they intersect, both systems in all other respects being arbitrary.

COR. 5°. *If A, B, C be the three centres, and AR, BS, CT the three radii, of any three circles, L, M, N the three radical axes of their three groups of two, O their radical centre, P and PQ the centre and radius of any fourth circle which intersects them, and α, β, γ the three angles of intersection; then—*

$$PL : PQ = BS.\cos\beta - CT.\cos\gamma : BC,$$

$$PM : PQ = CT.\cos\gamma - AR.\cos\alpha : CA,$$

$$PN : PQ = AR.\cos\alpha - BS.\cos\beta : AB.$$

For, if X, Y, Z be three of their six points of intersection with the fourth circle, and X', Y', Z' their three second points of intersection with its three radii PX, PY, PZ; then, since, by the fundamental property of the preceding article (182),

$$PY.PY' - PZ.PZ' = 2.BC.PL,$$

$$PZ.PZ' - PX.PX' = 2.CA.PM,$$

$$PX.PX' - PY.PY' = 2.AB.PN,$$

therefore, as in Cor. 6°. of the same,

$$PL : PQ = YY' - ZZ' : 2BC,$$
$$PM : PQ = ZZ' - XX' : 2CA,$$
$$PN : PQ = XX' - YY' : 2AB,$$

and since (62),

$$XX' = 2AR.\cos\alpha,$$
$$YY' = 2BS.\cos\beta,$$
$$ZZ' = 2CT.\cos\gamma,$$

therefore &c.

Hence, for the three circles whose centres are A, B, C and radii AR, BS, CT, the centre P of the circle which intersects them at the three angles α, β, γ lies on the line passing through their radical centre O which makes with the three radical axes L, M, N angles whose sines are proportional (61) to the three quantities

$$\frac{BS.\cos\beta - CT.\cos\gamma}{BC},\ \frac{CT.\cos\gamma - AR.\cos\alpha}{CA},\ \frac{AR.\cos\alpha - BS.\cos\beta}{AB}$$

and which therefore is given when the three circles and the three angles of intersection are given.

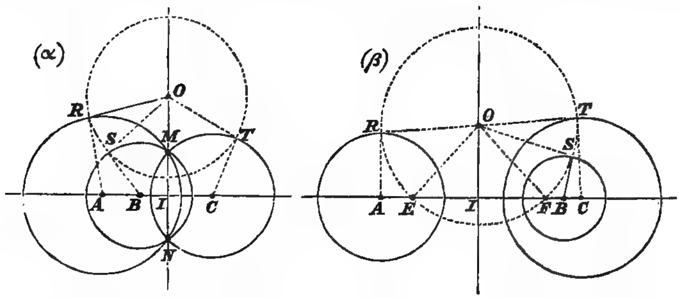
Cor. 6°. It appears, from the preceding, (Cor. 5°.), that the solution of the general problem "*to describe the circle which intersects three given circles at three given angles,*" is reduced to that of the problem "*to describe the circle having its centre on a given line and intersecting two given circles at given angles,*" which in the particular case of *contacts* of assigned species with the two (23), (to which, as we shall see in the sequel, every other case may be reduced), can always be solved by (54); the sum or difference, according to the nature of the contacts, of the distances of its centre from those of the two circles being evidently given. Of the celebrated problem "*to describe the circle having contacts of given species with three given circles,*" which is of course a particular case of the above, a more general and instructive solution will be given in the next chapter.

184. Any number of circles whose centers A, B, C, &c. are collinear, and whose radii AR, BS, CT, &c. are such that

$$AR^2 - AI^2 = BS^2 - BI^2 = CT^2 - CI^2 = \&c. = \pm k^2,$$

I being any point on the line of centres, are said to be *coaxal*, every two of them having evidently the same radical axis, viz. the perpendicular to the line of centres at the point I (181).

Of coaxal systems of circles there are two species, the sign of the constant difference or *modulus*, as it is termed, of the system $\pm k^2$ being positive for one and negative for the other; in the



former case, if M and N (fig. α) be the two points on the radical axis, for which

$$IM^2 = IN^2 = AR^2 - AI^2 = BS^2 - BI^2 = CT^2 - CI^2 = \&c. = k^2,$$

all the circles of the system (Euc. I. 47) pass evidently through M and N, and the system accordingly is said to be of *the common points species;* the two points M and N being common to all the circles, which, in that case, are all real, whatever be the positions of their centres A, B, C &c. upon the line on which they all lie; and, in the latter case, if E and F (fig. β) be the two points on the line of centers, for which

$$IE^2 = IF^2 = AI^2 - AR^2 = BI^2 - BS^2 = CI^2 - CT^2 \;\&c. = k^2,$$

all the circles of the system (Euc. II. 5, 6) have evidently E and F for a common pair of inverse points (149), and the system accordingly is said to be of *the limiting points species;* the two points E and F being evanescent limits (152) to the circles, which, in that case, are real or imaginary, according as their centres A, B, C, &c. are external or internal to the intercepted segment EF of the line on which they all lie. In both cases alike the radical axis itself is evidently the part not at infinity of the particular circle of the system corresponding to the particular position of the centre at an infinite distance, and is the common axis of reflexion (50) of the several pairs of evidently

equal circles of the system whose centres are equidistant in opposite directions from the central point I; for which particular point as centre the square of the radius of the corresponding circle of the system is evidently the absolute minimum in the former case, and the negative maximum in the latter.

In the particular case when the constant $k = 0$, the system may be regarded as belonging indifferently to either of the above species, or, more properly, as being at once in the limiting state of each, and at the transition phase from one to the other; the two common points M and N of the former species, or the two limiting points E and F of the latter, then evidently coinciding at the point I, the circles of the system all passing through that point, and the system itself being of the comparatively simple kind considered in (18).

It is evident from the above that two circles, given in magnitude and position, determine in all cases the coaxal system, whatever be its species, to which they belong; for, by the preceding article (181), they determine the position of the central point I, and with it the value of the modulus $\pm k^2$, of the system, and therefore &c.

185. Connected with every coaxal system of either species, as explained in the preceding article, there exists a conjugate system of the other species; the two common or limiting points of one being the two limiting or common points of the other; the radical and central axes of one being the central and radical axes of the other; the constant difference of squares or modulus for one being equal in magnitude and opposite in sign to the constant difference of squares or modulus for the other; and every circle of one system intersecting at right angles every circle of the other system; which latter property is evident from the consideration, that every circle coaxal (152) with the two common points M and N of a common points system, or passing through the two limiting points E and F of a limiting points system (see the dotted circles in figs. α and β of the preceding article) intersects at right angles, by (156), every circle of the system.

Between the original and its conjugate or orthogonal system, as in consequence of the preceding property it is also termed,

the relations, as above stated, are evidently reciprocal (8); either being transformable into the other by the simple interchange of the elements peculiar to its character, and every property true of either in relation to the other being, consequently, true also of the latter in relation to the former.

186. *Given two circles of a coaxal system of either species, to determine the circle of the system which,* 1°. *passes through a given point;* 2°. *cuts orthogonally a given line or circle;* 3°. *touches a given line or circle.*

These three problems, to which many others in the theory of coaxal circles are reducible, require different solutions according as the system to which the given circles belong is of the common or of the limiting points species; in the former case, the two points common to both on their radical axis are the common points of the system, and in the latter case, the two points inverse to both on their central axis (155, Cor. 2°.) are the limiting points of the system; and the common or limiting points, as the case may be, being thus given, the solutions, based in the latter case on the general property of the preceding article, are respectively as follows:

To solve 1°.; in the former case, the circle passing through the given point and through the two common points is that required; and in the latter case, the tangent at the given point to the circle passing through it and through the two limiting points intersects the central axis at the centre of the required circle (152). To solve 2°.; in the former case, the circle passing through the two common points and through the inverse of either with respect to the line or circle is that required (156); and in the latter case, the two circles passing through the two limiting points and touching the line or circle (51) determine on the latter its two points of intersection with the required circle. And to solve 3°.; in the former case, the two circles passing through the two common points and touching the line or circle (51) are those required; and in the latter case, the circle passing through the two limiting points and through the inverse of either with respect to the line or circle determines on the latter its points of contact with the two circles required.

187. For coaxal systems in general, whatever be their species, it is evident, from Cors. 1°, 3°, 4°, 5°, Art. 182, that—

1°. The tangents, real or imaginary, to all the circles of a coaxal system from any point on their radical axis are equal; and, conversely, when three or more circles are such that for two points, not at infinity, the tangents to them, real or imaginary, are equal, they are coaxal, and the line containing the two points is their radical axis.

2°. The chords of intersection, real or imaginary, of all the circles of a coaxal system with any arbitrary circle concur to a point on their radical axis; and, conversely, when three or more circles are such that their chords of intersection, real or imaginary, with two others are concurrent, they are coaxal, and the line containing the two points of concurrence is their radical axis.

3°. Every circle having its centre on the radical axis intersects all the circles of a coaxal system at angles, of the same affection, whose cosines are inversely as their radii; and, conversely, when three or more circles are intersected by two others at angles, of the same affection, whose cosines are inversely as their radii, they are coaxal, and the line of centres of the two is their radical axis.

4°. Every circle having its centre on the radical axis and intersecting any circle of a coaxal system at right angles intersects every circle of the system at right angles; and, conversely, when three or more circles intersect two others at right angles, they are coaxal, and the line of centres of the two is their radical axis.

It is evident also from (177), combined with the preceding property 4°, that—

5°. Every point has a common conjugate with respect to all the circles of a coaxal system, viz. the diametrically opposite point of the circle of the orthogonal system which passes through it; and, conversely, when three or more circles have two pairs of common conjugate points, whose distances are not at once equal and concentric, they are coaxal, as intersecting two different circles at right angles (4°).

188. For systems of the limiting points species in particular, it is also evident, from the properties referred to, that—

1°. The two limiting points are inverse points with respect to every circle of the system; and, conversely, when two circles have a common pair of inverse points, those points are the limiting points of the coaxal system they determine (152).

2°. Each limiting point and the perpendicular to the line of centres passing through the other are pole and polar with respect to every circle of the system; and, conversely, when two circles have a common pole and polar, the pole and polar centre are the limiting points of the coaxal system they determine (165).

3°. The tangents to every circle of the system from each limiting point are bisected by the radical axis; and, conversely, when the tangents to two circles from a point on their line of centers are bisected by their radical axis, that point is a limiting point of the coaxal system they determine (157).

4°. The tangents to every circle of the system from any point in the radical axis are equal to the distances of the point from the two limiting points; and, conversely, when the tangents to two circles from any point in their radical axis are equal to the distances of the point from two points on their line of centres, the latter are the limiting points of the coaxal system they determine (157).

5°. Every circle passing through the two limiting points is orthogonal to every circle of the system; and, conversely, when two circles which do not intersect are orthogonal to two circles which do, the common points of the coaxal system determined by the intersecting pair are the limiting points of the coaxal system determined by the non-intersecting pair (156).

6°. For every line touching two circles of the system, the segment intercepted between the points of contact subtends a right angle at each limiting point; and, conversely, when for a line touching two circles the segment intercepted between t' · points of contact subtends right angles at two points on their line of centres, those points are the limiting points of the coaxal system they determine. (22, 1', and Euc. III. 31.)

189. *If X, Y, Z be any three collinear points on the three sides BC, CA, AB of any triangle ABC.*

1°. *The three circles on the three connectors AX, BY, CZ, as diameters, are coaxal.*

2°. *The four polar centres of the four triangles YAZ, ZBX, XCY, and ABC are on their radical axis.*

3°. *The four polar circles of the four triangles YAZ, ZBX, XCY, and ABC are coaxal.*

4°. *The three middle points of the three connectors AX, BY, CZ are on their radical axis.*

For, the three connectors AX, BY, CZ being three lines from the vertices to the opposite sides of each of the four triangles YAZ, ZBX, XCY, and ABC; therefore, by Cors. 3°. and 1°. Art. 183, the four polar circles of the four triangles (168) intersect at right angles the three circles of which the three connectors are diameters, and, as consequently the circles of the two groups are conjugately coaxal (185), therefore &c.

Cor. 1°. The four lines BXC, CYA, AZB, and XYZ in the above being entirely arbitrary, the four properties consequently may be stated, otherwise thus, as follows—

1°. *The three circles of which the three chords of intersection of any four lines are diameters are coaxal.*

2°. *The four polar centres of the four triangles determined by the four lines are on their radical axis.*

3°. *The four polar circles of the four triangles determined by the four lines are coaxal.*

4°. *The three middle points of the three chords of intersection of the four lines are on their radical axis.*

Cor. 2°. The centres of all circles of a coaxal system being collinear (184), and the two lines of centres of two conjugate systems being orthogonal (185), it follows, consequently, from Cor. 1°., that, for every system of four arbitrary lines—

1°. *The three middle points of the three chords of intersection they determine are collinear.*

2°. *The four polar centres of the four triangles they determine are collinear.*

3°. *The two lines of collinearity for the middle points of the chords and for the polar centres of the triangles are orthogonal.*

The preceding properties may be established by many other considerations, but by none more simply or elegantly than the above, which are due to Mr. W. F. Walker.

190. *If D, E, F be three circles connected with three others*

A, B, C by the relations that D is coaxal with B and C, E with C and A, and F with A and B, then—

1°. *They have always the same radical centre and product with A, B, C.*

2°. *When they pass through a common point P they pass through a second common point P'.*

3°. *When their centres are collinear they are themselves coaxal.*

For, if RR', SS', TT' be any three chords of A, B, C concurring to their radical centre O, and UU', VV', WW' any three of D, E, F concurring to the same point O; then, to prove 1°., since by hypothesis, the three groups of circles B, C, and D; C, A, and E; A, B, and F are coaxal, therefore (187, 1°) the three groups of rectangles $OS.OS'$, $OT.OT'$, and $OU.OU'$; $OT.OT'$, $OR.OR'$, and $OV.OV'$; $OR.OR'$, $OS.OS'$, and $OW.OW'$ are equal in magnitude and sign, and therefore the two groups of circles D, E, F and A, B, C have the same radical centre and product (183, Cor. 1°.); to prove 2°., when D, E, F pass through a common point P they pass also through a second common point P', that viz. on the line OP for which the product $OP.OP'$ is equal in magnitude and sign to their radical product, and of course to that of A, B, C (1°.); and to prove 3°., when the centres of D, E, F are collinear, if X, Y, Z be their three centres, XU, YV, ZW their three radii, and I the foot of the perpendicular from O on the line XYZ; since then, by 1°,

$$XO^2 - XU^2 = YO^2 - YV^2 = ZO^2 - ZW^2,$$

therefore (Euc. I. 47),

$$XI^2 - XU^2 = YI^2 - YU^2 = ZI^2 - ZW^2,$$

and therefore the three circles D, E, F are coaxal (184).

Otherwise thus: the circle G orthogonal to the three A, B, C being, by (182, Cor. 5°.), orthogonal also to the three D, E, F, its centre O and the square of its radius OG^2 are, by (183, Cor. 1°.), the radical centre and product of both triads A, B, C and D, E, F which proves 1°.; when D, E, F pass through a common point P, they pass also, by (156), through its inverse P' with respect to the circle G, which proves 2°.; and when the centres of D, E, F are collinear, they are at once orthogonal to the circle G and to the line of their centres (22, 1°.), and therefore coaxal (187, 4°.), which proves 3°.

COR. In the general case, if P and P', Q and Q', R and R' be the three pairs of intersections of the three pairs of circles E and F, F and D, D and E, and if X, Y, Z be the three intersections of the three pairs of lines QR and $Q'R'$, RP and $R'P'$, PQ and $P'Q'$; then since, by 1°, the three lines PP', QQ', RR' are concurrent, and the three rectangles $OP.OP'$, $OQ.OQ'$, $OR.OR'$ are equal in magnitude and sign, therefore (140) the three points X, Y, Z are collinear, and, (Euc. III. 35, 36), the three pairs of rectangles $XQ.XR$ and $XQ'.XR'$, $YR.YP$ and $YR'.YP'$, $ZP.ZQ$ and $ZP'.ZQ'$ are equal in magnitude and sign; or, in other words, the two triangles PQR and $P'Q'R'$ are in perspective, and the centre and axis of their perspective are the radical centre of the three circles A, B, C and the radical axis of the two PQR and $P'Q'R'$.

191. *If A, B, C be the three centres, and AR, BS, CT the three radii, of any three coaxal circles, the relation*

$$\frac{AR^2}{AB.AC}+\frac{BS^2}{BC.BA}+\frac{CT^2}{CA.CB}=1$$

is true in all cases, whatever be the species of the systems to which they belong.

For since, by hypothesis (184), I being the central point of the system, $AR^2-AI^2=BS^2-BI^2=CT^2-CI^2=\pm k^2=$ the modulus of the system; therefore,

$$BC.AR^2+CA.BS^2+AB.CT^2$$
$$=BC.AI^2+CA.BI^2+AB.CI^2\pm(BC+CA+AB).k^2;$$

but, by (78), $BC+CA+AB=0$, and, by (83),

$$BC.AI^2+CA.BI^2+AB.CI^2=-BC.CA.AB,$$

therefore

$$BC.AR^2+CA.BS^2+AB.CT^2=-BC.CA.AB,$$

and therefore &c., the latter relation being evidently equivalent to the above.

The above general relation, which when the circles belong to a system of the common points species is evident from (83), may be regarded as *the criterion of coaxality* between three circles whose centres are collinear, and by aid of it the radius corresponding to a given centre, of any circle coaxal with two

others, is given at once without requiring the previous determination of the central point I and of the modulus $\pm k^2$ of the system; it is evident also from it that when two of three coaxal circles are concentric and unequal, the third, as it ought (181, 4°.), is concentric with both.

COR. 1.° If $CT=0$, that is, if, in a system of the limiting points species, C be one of the two limiting points; then, whatever be the positions of A and B and the magnitudes of AR and BS,

$$\frac{AR^2}{AC} - \frac{BS^2}{BC} = AB,$$

which accordingly is the relation by which to calculate in numbers the positions, real or imaginary, of the two limiting points, when two circles of the system are given in magnitude and position.

COR. 2°. If $AR=0$ and $BS=0$, that is, if, in a system of the limiting points species, A and B are the two limiting points; then, for every position of C, whatever be the interval AB,

$$CT^2 = CA.CB,$$

from which it appears, as stated in (184), that, in a system of the limiting points species, the two limiting points are inverse points with respect to every circle of the system (152).

192. *If A, B, C be the centres of three coaxal circles, AR, BS, CT their three radii, and PR, PS, PT the three tangents to them from any point P not at infinity, the relation*

$$BC.PR^2 + CA.PS^2 + AB.PT^2 = 0$$

is true in all cases, whatever be the species of the system to which they belong.

For, since, by the general relation of Art. 83,

$$BC.AP^2 + CA.BP^2 + AB.CP^2 = -BC.CA.AB;$$

and since, by the general relation of the preceding article,

$$BC.AR^2 + CA.BS^2 + AB.CT^2 = -BC.CA.AB;$$

therefore, at once, by subtraction,

$$BC.(AP^2 - AR^2) + CA.(BP^2 - BS^2) + AB.(CP^2 - CT^2) = 0,$$

which is manifestly the same as the above.

Otherwise thus: if D be the centre of the circle of the system which passes through P, then since, by Cor. 2°, Art. 182,

$$PR^2 = 2.AD.LP, \quad PS^2 = 2.BD.LP, \quad PT^2 = 2.CD.LP;$$

therefore, multiplying by BC, CA, AB, and adding

$$BC.PR^2 + CA.PS^2 + AB.PT^2$$
$$= 2.LP.(BC.AD + CA.BD + AB.CD) = 0,$$

since LP by hypothesis is not $= \infty$, and therefore &c.

Cor. 1°. If $PT = 0$, that is, if P be on the circle C, then

$$BC.PR^2 + CA.PS^2 = 0, \text{ or } PR^2 : PS^2 = AC : BC,$$

and, conversely, if the latter relation exist, then $PT = 0$, or P is on the circle C. Hence—

When three circles are coaxal, the squares of the tangents to two of them from any point on the third have the constant ratio of the distances of their centres from the centre of the third; and, conversely, the locus of a variable point the squares of the tangents from which to two fixed circles have any constant ratio, positive or negative, is the coaxal circle whose centre divides the distance between their centres in the magnitude and sign of the ratio.

By aid of Cor. 2°, Art. 182, this important property, which obviously includes those of Art. 158, and of Cor. 1°, Art. 182, as particular cases, may be proved, otherwise thus, as follows: since, by the corollary in question, when P lies on the circle C, then $PR^2 = 2.AC.LP$, $PS^2 = 2.BC.LP$, and conversely, therefore, at once, by division, $PR^2 : PS^2 = AC : BC$, and therefore &c.

Cor. 2°. If $PR = 0$, and $PS = 0$, that is, if P be on two of the circles A and B at once, then $PT = 0$, or P is on the third circle also. Hence, as already stated in (184), when two circles intersect, every third circle coaxal with them passes through their points of intersection.

Cor. 3°. *If M and N be the two points of contact with any line of the two circles of any coaxal system which touch it, P and Q its two points of intersection with any third circle of the system, and O its point of intersection with the radical axis, then always*

$$PM^2 : QM^2 = PN^2 : QN^2 = PO : QO.$$

For, if A and B be the centres of the two circles touching the line at M and N, C that of the circle intersecting it at P and Q, and L the radical axis of the system; then since, as above, by Cor. 2°, Art. 182,

$$PM^2 = 2.AC.PL, \quad QM^2 = 2.AC.QL,$$
$$PN^2 = 2.BC.PL, \quad QN^2 = 2.BC.QL;$$

therefore, at once, by division,

$$PM^2 : QM^2 = PN^2 : QN^2 = PL : QL,$$

and since, by similar triangles, $PL : QL = PO : QO$, therefore &c.

Cor. 4°. *In the same case, for a system of the limiting points species, if E and F be the two limiting points, the two angles MEN and MFN are right angles, and their sides bisect externally and internally the two angles PEQ and PFQ respectively*: see 186, 6°.

For, since by Cor. 1°,

$$PM^2 : PN^2 : PE^2 : PF^2 = QM^2 : QN^2 : QE^2 : QF^2,$$

therefore at once, by alternation,

$$PM^2 : QM^2 = PN^2 : QN^2 = PE^2 : QE^2 = PF^2 : QF^2,$$

and therefore &c. (Euc. VI. 3.)

Cor. 5°. *If P, Q, R be the three vertices of any triangle inscribed in any circle of a coaxal system, X, Y, Z the three external, and X', Y', Z' the three internal, points of contact with its sides QR, RP, PQ of the six circles of the system which touch them in pairs* (186, 3°), *then always—*

a. The four groups of three points Y', Z', X; Z', X', Y; X', Y', Z; and X, Y, Z are collinear.

b. The four groups of three lines QY, RZ, PX'; RZ, PX, QY'; PX, QY, RZ'; and PX', QY', RZ' are concurrent.

For, if A and A', B and B', C and C' be the centers of the six circles touching QR, RP, PQ at X and X', Y and Y', Z and Z' respectively, and D the centre of the circle containing P, Q, R, then since, by Cor. 1°,

$$\frac{PY^2}{PZ^2} = \frac{BD}{CD}, \quad \frac{QZ^2}{QX^2} = \frac{CD}{AD}, \quad \frac{RX^2}{RY^2} = \frac{AD}{BD},$$

with seven other groups of the same form, one for the accented,

and six for the mixed accented and unaccented letters; therefore, at once, by composition of ratios,

$$\frac{PY^2}{PZ^2}\cdot\frac{QZ^2}{QX^2}\cdot\frac{RX^2}{RY^2}=1,$$

and similarly for each of the seven remaining groups, and therefore &c.

It is evident also, from Cor. 3°, that the three intercepts XX', YY', ZZ' between the points of contact of the three pairs of circles touching the three sides of the triangle, are cut internally and externally in common ratios by every circle of the system, and are bisected internally by its radical axis.

Cor. 6°. The general relation of the present article may obviously be stated in the equivalent form

$$BC.PX.PX'+CA.PY.PY'+AB.PZ.PZ'=0,$$

X and X', Y and Y', Z and Z', being the pairs of intersections with the three circles of any three lines passing through P. This form has the comparative advantage, that the three rectangles it involves, whatever be their signs, are always real, whereas the three tangents, whose squares are involved in the original form, may be, and often are, some or all, imaginary.

193. *If A, B, C be the centres of three coaxal circles, AR, BS, CT their three radii, and α, β, γ their three angles of intersection with any arbitrary circle whose centre is not at infinity, the relation*

$$BC.AR.\cos\alpha+CA.BS.\cos\beta+AB.CT.\cos\gamma=0$$

is true in all cases, whatever be the species of the system to which they belong.

For, if P be the centre of the arbitrary circle, PQ its radius, X, Y, Z three of its six points of intersection with the three coaxal circles, and X', Y', Z' their three second points of intersection with its three radii PX, PY, PZ; then since, by the general property of the preceding article,

$$BC.PX.PX'+CA.PY.PY'+AB.PZ.PZ'=0;$$

and since in the present case $PX=PY=PZ=PQ$, therefore

$$(BC+CA+AB).PQ^2+(BC.XX'+CA.YY'+AB.ZZ').PQ=0,$$

from which as $BC+CA+AB=0$, and as PQ not $=\infty$, therefore

$$BC.XX'+CA.YY'+AB.ZZ'=0,$$

which, as

$$XX'=2.AR.\cos\alpha,\ \ YY'=2.BS.\cos\beta,\ \ ZZ'=2.CT.\cos\gamma,$$

is therefore equivalent to the above.

Otherwise thus: by Cor. 6°, Art. 182, see also Cor. 5°, Art. 183,

$$\frac{BS.\cos\beta-CT.\cos\gamma}{BC}=\frac{CT.\cos\gamma-AR.\cos\alpha}{CA}=\frac{AR.\cos\alpha-BS.\cos\beta}{AB},$$

each being $=PL:PQ=$ the cosine of the angle, real or imaginary, Cor. 8°, Art. 182, at which the arbitrary circle intersects the radical axis L of the three coaxal circles A, B, C, and from either of these equalities the above manifestly results immediately.

This latter method has the advantage over the former, of not only establishing the general relation connecting the cosines of the three angles of intersection, real or imaginary, of any arbitrary circle with three coaxal circles, but of connecting with them at the same time the cosine of its angle of intersection, real or imaginary, with their radical axis.

Cor. 1°. When C is such, that

$$BC.AR.\cos\alpha=AC.BS.\cos\beta;$$

or, which is the same thing, that

$$AC:BC=AR.\cos\alpha:BS.\cos\beta,$$

then $CT.\cos\gamma=0$, and therefore $\cos\gamma=0$; except only when $CT=0$, in which case it is indeterminate. Hence—

Every circle intersecting two circles A and B at two angles α and β intersects at right angles the coaxal circle C whose centre is given by the preceding proportion; except only when that circle is a point, in which case it passes through it, and intersects it at an indeterminate angle.

Cor. 2°. When in Cor. 1°, $\cos\alpha:\cos\beta=\pm 1$, that is, when α and β are equal or supplemental, then $AC:BC=\pm AR:BS$, and therefore (44) C is a centre of similitude, external in the former case and internal in the latter case, of the circles A and B. Hence—

Every circle intersecting two circles A and B at equal or supplemental angles, intersects at right angles the coaxal circle, real or imaginary, whose centre is the external or internal centre of similitude of A and B; except only when that circle is a point, in which case it passes through it, and intersects it at an indeterminate angle.

Cor. 3°. When in Cor. 1°, $AR.\cos\alpha : BS.\cos\beta = \pm 1$, that is, when $\cos\alpha : \cos\beta = \pm BS : AR$, then $AC : BC = \pm 1$, and therefore C is a point of bisection, external in the former case and internal in the latter case, of the interval AB. Hence—

Every circle intersecting two others A and B at angles, of similar or opposite affections, whose cosines are inversely as their radii, intersects at right angles the coaxal circle whose centre bisects externally or internally the interval between the centres of A and B; except only, in the latter case, when that circle is a point, in which case it passes through it, and intersects it at an indeterminate angle.

Cor. 4°. When $AR.\cos\alpha = 0$, that is, when either $AR = 0$ or $\cos\alpha = 0$, then $AC.BS.\cos\beta = AB.CT.\cos\gamma$, or, as before (Cor. 1°.), $BS.\cos\beta : CT.\cos\gamma = BA : CA$. Hence—

Every circle either passing through a point or cutting orthogonally a line or circle A, and intersecting another line or circle B at any other constant angle β, intersects every third line or circle C coaxal with A and B at a third constant angle γ, whose cosine is given by the preceding relation.

Cor. 5°. When $AR.\cos\alpha = 0$, and $BS.\cos\beta = 0$, that is, when either $AR = 0$ or $\cos\alpha = 0$, and either $BS = 0$ or $\cos\beta = 0$, then $CT.\cos\gamma = 0$, whatever be the position of C, and therefore $\cos\gamma = 0$; except only when $CT = 0$, in which case it is indeterminate. Hence, see 156 and 185—

Every circle passing through two points, or cutting orthogonally two circles, or passing through a point and cutting orthogonally a circle, cuts orthogonally every circle coaxal with the two; except only when that circle is a point, in which case it passes through it, and intersects it, like every other evanescent circle, at an indeterminate angle.

Cor. 6°. When C is such, that

$$BC.AR.\cos\alpha - AC.BS.\cos\beta = \pm AB.CT,$$

then $\cos\gamma = \mp 1$, and therefore γ either = two right angles or = 0. Hence—

Every circle intersecting two circles A and B at two angles α and β touches, one externally and one internally, the two coaxal circles whose centres are given by the preceding relation.

Cor. 7°. When $\cos\alpha = \pm 1$, and $\cos\beta = \pm 1$, that is, when α either = 0 or = two right angles, and β either = 0 or = two right angles, then $AB.CT.\cos\gamma = \mp BC.AR \pm AC.BS$. Hence—

Every circle touching, with definite contacts, two circles A and B intersects any coaxal circle C at the angle γ whose cosine is given by the preceding relation.

Cor. 8°. In general, when two of the circles A and B and the two corresponding angles of intersection α and β are given, then, in virtue of the general relation, the third circle C determines the third angle γ, and conversely. Hence, generally—

Every circle intersecting two circles A and B at the same two angles α and β, intersects every third circle C coaxal with them at the same third angle γ determined by the general relation, cuts orthogonally the particular circle D determined by the relation Cor. 1°, *and touches, one internally and one externally, the two particular circles E and F determined by the relation* Cor. 6°.

The two circles A and B and the two angles α and β being given, to determine the two circles E and F coaxal with A and B which are touched by the intersecting circle in every position; describing *any* circle K intersecting A and B at the given angles α and β, the two circles E and F coaxal with A and B which touch the circle K (186, 3°), by the above, are those required.

Of the many circles K which could be described intersecting A and B at the given angles α and β, one of *given* radius is that most easily constructed; for when a circle of given radius intersects two given circles at given angles, its centre lies evidently on two concentric circles of given radii, and is therefore given.

It is evident that when one of the two intersected circles A is evanescent, then one of the two enveloped circles E coincides with it; and, that when the two intersected circles A and B are evanescent, then the two enveloped circles E and F coincide with them.

COR. 9°. By aid of the general property Cor. 8°—the general problem "*to describe a circle intersecting three given circles A, B, C at three given angles α, β, γ*" may be readily reduced to the particular case of itself: "*to describe a circle having contacts of assigned species with three given circles.*" For the required circle to intersect the three circles A, B, C at the three angles α, β, γ must touch with opposite contacts three pairs of circles D' and D'', E' and E'', F' and F'', coaxal with B and C, C and A, A and B respectively, and determinable by Cor. 8°; any three of these six, for different pairs of the given circles, being constructed by Cor. 6°, the circle touching them with the species of contact to which they correspond is that required.

By supposing first one and then two of the three given circles A, B, C to become evanescent, the two problems "*to describe a circle passing through a given point and intersecting two given circles at given angles,*" and "*to describe a circle passing through two given points and intersecting a given circle at a given angle,*" are obviously included in the above as particular cases.

COR. 10°. As, by the same general property Cor. 8°, the circle intersecting three given circles A, B, C at three given angles α, β, γ cuts orthogonally three circles D, E, F coaxal with B and C, C and A, A and B respectively, and determinable very readily by Cor. 1°; and as the problem to describe the circle orthogonal to three others is one that presents no difficulty (183, Cor. 1°); it might at first sight appear as if an easier solution of the general problem Cor. 9°, would be obtained by substituting the three auxiliary circles D, E, F in place of the three employed in the construction actually given; such, however, would not be the case, the three circles D, E, F being, as may be easily shewn, coaxal, and consequently admitting of an infinite number of orthogonal circles (185).

For, A, B, C being the three centres, and AR, BS, CT the three radii of the three given circles; if X, Y, Z be the three centres of the three circles D, E, F, then since, by Cor. 1°,

$$\frac{BX}{CX} = \frac{BS.\cos\beta}{CT.\cos\gamma}, \quad \frac{CY}{AY} = \frac{CT.\cos\gamma}{AR.\cos\alpha}, \quad \frac{AZ}{BZ} = \frac{AR.\cos\alpha}{BS.\cos\beta},$$

therefore, at once, by composition,

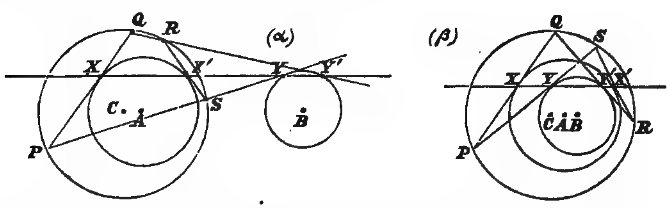
$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1,$$

consequently (134, a) the three centres X, Y, Z are collinear, and therefore (190, 3°) the three circles D, E, F are coaxal.

The general problem itself, "to describe the circle intersecting three given circles at three given angles," is of course, for the same reason, indeterminate or impossible when the circles are coaxal; indeterminate where the circles and angles are such as to fulfil the general relation of the present article, impossible when they are not.

194. With the two following converse properties of coaxal circles, and a few of the consequences to which they lead, we shall close the present Chapter.

When four points on two circles are collinear, the four vertices of the quadrilateral of which the tangents at the two on each are opposite sides lie on a third circle coaxal with both; and, conversely, when of a quadrilateral inscribed in a circle two opposite sides touch a second circle, the remaining two touch a third circle coaxal with the first and second, and the four points of contact with the two circles touched are collinear.



To prove the first part: If X and X', Y and Y' (figs. α and β) be the two pairs of points, A and B the centres of the two circles, α and β their two angles of intersection with the line of the points, and P, Q, R, S the four vertices of the quadrilateral, that is, the four points in which the two tangents at X and X' to one circle intersect the two at Y and Y' to the other; then since, in the four triangles XPY, XQY', $X'RY'$, $X'SY$, the four ratios $PX : PY$, $QX : QY'$, $RX' : RY'$, $SX' : SY$

are equal to the same ratio $\sin\beta : \sin\alpha$ (63), they are equal to each other, and therefore (192, Cor. 1°) the four points P, Q, R, S lie on the circle coaxal with A and B whose centre C is given by the relation $AC : BC = \sin^2\beta : \sin^2\alpha$; and therefore &c.

To prove the second part: If P, Q, R, S (same figures) be the four vertices of the quadrilateral, C the centre of the circle on which they lie, X and X' the points of contact of its pair of opposite sides PQ and RS with the second circle, A the centre of that circle, Y and Y' the points of intersection of the line XX' with the other pair of opposite sides PS and RQ of the quadrilateral, and B the centre of the circle which (Euc. III. 21, 22) touches those sides at Y and Y'; then since, by the first part, the circle C is coaxal with the two A and B, therefore, reciprocally, either of the latter is coaxal with the other and C, and therefore &c.

As to the possibility of a circle touching PS and RQ at Y and Y', it is evident, from Euc. III. 21, 22, that for the three pairs of angles of intersection α and α', β and β', γ and γ' of any line with the three pairs of opposite connectors L and L', M and M', N and N' of any four points P, Q, R, S on a circle, if $\alpha = \alpha'$ then $\beta = \beta'$ and $\gamma = \gamma'$, and therefore &c.

Cor. 1°. In both the above properties, while the three circles A, B, C remain fixed, the line and quadrilateral may vary simultaneously, provided only the ratio $\sin\alpha : \sin\beta$, of which, by the above, the ratio $BC : AC$ is the duplicate, be constant; or, which is the same thing, provided the ratio $XX' : YY'$ be constant, since (62) $XX' = 2.AX.\sin\alpha$, and $YY' = 2.BY.\sin\beta$. Hence—

If a variable line intersect two fixed circles at angles whose sines have any constant ratio, or, which is the same thing, intercept in them chords having any constant ratio, the four vertices of the quadrilateral, of which the tangents at the points of intersection with each are opposite sides, lie on the fixed circle coaxal with both whose centre divides the distance between their centres in the inverse duplicate of the constant ratio of the sines.

And, conversely—

If from a variable point on one of three fixed coaxal circles pairs of tangents be drawn to the other two, the four lines containing a point of contact with one and a point of contact with

the other intersect them at angles the squares of whose sines have the constant inverse ratio of the distances of their centres from the centre of the first, and therefore intercept in them chords whose squares divided by the squares of their radii have the same constant ratio.

In the particular case, when $\sin\alpha : \sin\beta = 1$, or (62) when $XX' : YY' = AX : BY$, or (44) when the line of intersection passes through a centre of similitude, external or internal, of the circles intersected, then $AC : BC = 1$, and therefore, of the four vertices of the quadrilateral $PQRS$, two opposites lie on the line at infinity, and the remaining two lie on the radical axis of the circles A and B, the two lines into which the coaxal circle C then breaks up (184); and the same is evident from the consideration that when $\alpha = \beta$ the pairs of tangents at two pairs of intersections X and Y, X' and Y' are parallel, and intersect consequently at infinity, and the pairs of tangents at the remaining pairs of intersections X and Y', X' and Y form isosceles triangles with the line of intersection and intersect consequently on the radical axis of A and B. (182, Cor. 1°.)

Cor. 2°. *If L and L', M and M', N and N' be the three pairs of opposite lines connecting any four points P, Q, R, S on a circle, X and X', Y and Y', Z and Z' their three pairs of intersections with any line making equal angles $\alpha = \alpha'$ with one pair of them LL', and therefore* (Euc. III. 21, 22) *pairs of equal angles $\beta = \beta'$ and $\gamma = \gamma'$ with the remaining two pairs MM' and NN'; the three circles touching L and L', M and M', N and N' at X and X', Y and Y', Z and Z' are coaxal with each other and with the circle $PQRS$.*

For, by the first part of the above, the latter circle is coaxal with every two of them, and therefore &c.

If the intersecting line pass, as it or a parallel to it in every case may, through one of the three points LL', MM', NN', the corresponding circle of contact being then evanescent, that point is consequently a limiting point of the coaxal system to which the remaining two and the circle $PQRS$ belong; and if it pass through two of them at once, which, in compliance with the condition restricting it to one or other of two rectangular directions, it only could do when one of the three is at infinity, the two corresponding circles of contact being then evanescent, these points are consequently the two limiting points of the

coaxal system to which the third and the circle $PQRS$ belong. Hence, see Cor. 2°, Art. 163, *the two centres of perspective of any two parallel chords of a circle are at once inverse points with respect to the circle itself and to that which touches the two chords at their middle points;* a property the reader may easily verify, *a priori*, for himself.

Cor. 3°. *If X and X', Y and Y', Z and Z' be the three pairs of intersections of an arbitrary line with any three circles, L and L', M and M', N and N' the three pairs of tangents at them to the circles; the three circles containing the vertices of the three quadrilaterals, of which MM' and NN', NN' and LL', LL' and MM' are pairs of opposite sides, are coaxal.*

For, if A, B, C be the centres of the three original circles, α, β, γ their three angles of intersection with the line, and A', B', C' the centres of the three circles containing the vertices of the three quadrilaterals, which, by the above, are coaxal with the pairs of the originals whose centres are B and C, C and A, A and B respectively; then since, by the above,

$$\frac{BA'}{CA'} = \frac{\sin^2\gamma}{\sin^2\beta}, \quad \frac{CB'}{AB'} = \frac{\sin^2\alpha}{\sin^2\gamma}, \quad \frac{AC'}{BC'} = \frac{\sin^2\beta}{\sin^2\alpha};$$

therefore, at once, by composition of ratios,

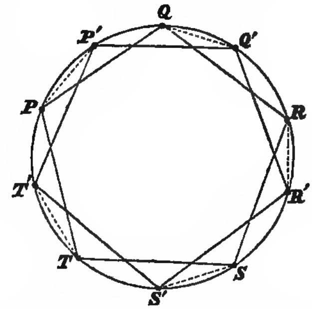
$$\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = 1,$$

consequently (134, a) the three centres A', B', C' are collinear, and therefore &c., (190, 3°.).

In the particular case when the centres of the three original circles A, B, C are collinear, those of the three derived circles A', B', C' are of course necessarily collinear with them; but the preceding relation, proved exactly as above, exists in the particular as in the general case, and equally in both establishes the coaxality of the derived system; the same remark applies to the similar property proved, in a similar manner, in Cor. 10° of the preceding Article.

Cor. 4°. *For a variable polygon of any order inscribed to a fixed circle of any coaxal system, if all the sides but one touch in every position fixed circles of the system, that one also touches in every position a fixed circle of the system.*

Let P, Q, R, S, &c. T, and P', Q', R', S', &c. T' be any two positions of the vertices of the polygon on the circle of the system round which they move. If in the two positions the several pairs of sides PQ and $P'Q'$, QR and $Q'R'$, RS and $R'S'$, &c. up to, but not including, the last, touch the same circles of the system, the last pair TP and $T'P'$ also touch the same circle of the system.



For, joining the extremities PP', QQ', RR', SS', &c. TT' of the several pairs of sides touching the same circles in the two positions of the polygon; then since, by hypothesis, PQ and $P'Q'$ touch a common circle of the system, therefore, by the second part of the above, PP' and QQ' touch a common circle of the system; since again, by hypothesis, QR and $Q'R'$ touch a common circle of the system, therefore again, by the same, QQ' and RR' touch a common circle of the system; since again, by hypothesis, RS and $R'S'$ touch a common circle of the system, therefore again, by the same, RR' and SS' touch a common circle of the system; and so on to the last pair of sides but one; from which it follows that the first and last connectors PP' and TT' touch a common circle of the system, and therefore, by the same as before, that the last pair of sides TP and $T'P'$ touch a common circle of the system.

This simple and elegant demonstration of the above celebrated Theorem of Poncelet is due to Dr. Hart, who published an extension of it in *The Quarterly Journal of Pure and Applied Mathematics*, Vol. II., page 143; a proof nearly identical was arrived at independently about the same time by Mr. Casey.

Cor. 5°. The principle of the above demonstration depending on the circumstance that the several chords of connection PP', QQ', RR', SS', &c. TT' for any two positions of the polygon all touch a common circle of the system, and that again depending only on the circumstance that every circle of the system touched by a side of the polygon in one position is touched also by a side of the polygon in the other position, irrespectively altogether of the circumstance as to whether the contacts of

the several sides of the polygon with the circles they touch take place in the same order of sequence in the two positions or not; hence, *in Poncelet's Theorem the order of sequence in which the several circles enveloped are touched by the several successive sides of the variable polygon in its different positions is entirely arbitrary, provided only no circle touched in one position be omitted in another;* a circumstance noticed by Poncelet himself, and established by him, in connection with the Theorem, on principles instructive and suggestive but involving conceptions beyond the limits of mere elementary geometry.

To see this clearly, the figure and notation employed above, for facility of conception in the first instance, being adapted only to the case when the order of the contacts with the several circles touched is the same in the two positions of the polygon; denoting by P_1P_2 and $P_1'P_2'$, Q_1Q_2 and $Q_2'Q_2'$, R_1R_2 and $R_1'R_2'$, S_1S_2 and $S_1'S_2'$, &c., the several pairs of sides of the two polygons corresponding to the two positions which touch the same circles A, B, C, D, &c. of the system, measured cyclically in the same direction for each polygon (110), but in similar or opposite directions for both, and independently altogether of the order of sequence in either; then since, by the above, the several pairs of connecting chords P_1P_1' and P_2P_2', Q_1Q_1' and Q_2Q_2', R_1R_1' and R_2R_2', S_1S_1' and S_2S_2', &c. touch the same circles of the system A', B', C', D', &c.; and since, from the nature of the case (every side of a polygon being conterminous with the two adjacent), every connector of the system P_1P_1', Q_1Q_1', R_1R_1', S_1S_1', &c. coincides necessarily with some connector of the opposite system P_2P_2', Q_2Q_2', R_2R_2', S_2S_2', &c., and conversely; therefore the several circles A', B', C', D', &c. touched by the several pairs of connectors all coincide, and therefore &c.

Cor. 6°. It appears at once from the above, Cors. 4° and 5°, that the general problem, "*to inscribe in a given circle of a coaxal system a polygon of any degree whose several sides in any order of sequence shall touch given circles of the system,*" is indeterminate when the circles are such that for every polygon inscribed to the first, all whose sides but one touch in any order of sequence all the others but one, the last side touches the last circle; when this is not the case the four common tangents, real or imaginary, to the last circle and to that touched in every

position by the last side (Cor. 4°), give the last sides of the four polygons that solve the problem, and with them therefore the polygons themselves.

Since, when two circles intersect, two of their four common tangents, those passing through their external centre of similitude, are always real, and the other two, those passing through their internal centre of similitude, are always imaginary; hence when, in the above problem, the coaxal system to which the circles belong is of the common points species, two of the four polygons that solve it are always real and the other two always imaginary; when, however, the system is of the limiting points species, all four may be real or all four imaginary according to circumstances.

Cor. 7°. As all the circles touched by the several sides of the variable polygon in every position may coincide, thus reducing the several circles in the general case to two, it appears therefore, from the same, that the modified problem, "*to construct a polygon of any order all whose vertices shall lie on one given circle and all whose sides shall touch another given circle,*" is indeterminate when the two circles are such that for every polygon of the required order all whose vertices lie on the first, and all whose sides but one touch the second, the last side also touches the second. When this is not the case the four common tangents, real or imaginary, to the second circle, and to the third circle, coaxal with the first and second, which is touched in every position by the last side (Cor. 4°), give, as in Cor. 6°, the last sides of the four polygons that solve the problem; which polygons for all *odd* orders, by taking the two symmetrical positions for which the last side is perpendicular to the line of centres of the three circles, are easily seen to be all real, all imaginary, or, two real and two imaginary, according as the distance between the centres of the two given circles is greater than the sum, less than the difference, or, intermediate between the sum and difference, of their radii.

In the particular case when the polygon is a triangle, the condition for indeterminateness, as regards the centres and radii of the two given circles, is given immediately by the known relation (102, Cor. 4°) that for every triangle, having no exceptional peculiarity of form, the square of the distance between

the centres of the circle passing through its three vertices and of any of the four touching its three sides = the square of the radius of the former ± twice the rectangle under the radii of both; when, therefore, for two circles given in magnitude and position, the centres and radii fulfil either condition expressed in that relation, the problem to construct a triangle having its three vertices points on one and its three sides tangents to the other is indeterminate; and when they do not, though four or two real solutions still exist under the circumstances stated above, the resulting triangles, as may be easily seen on drawing the figures corresponding to the two cases, have each a pair of coincident sides, and therefore, besides their ordinary inscribed and exscribed circles, which for them as for every other triangle fulfil the relation, have each an indefinite number of other circles touching its three sides, which, including the given circle touched by the three, do not fulfil either relation. The fact as well as the explanation of the existence of real solutions in the latter case has hitherto been very generally overlooked by geometers.

CHAPTER XII.

ON THE CENTRES AND AXES OF PERSPECTIVE OF CIRCLES CONSIDERED IN PAIRS.

195. THE two points on the common diameter of two circles which divide the interval between their centres, externally and internally, in the ratio of the conterminous radii, are termed indifferently (44) the two *centres of similitude*, external and internal, and also (144) the two *centres of perspective*, external and internal, of the circles; that they possess a double right to the latter appellation will appear in the sequel.

From the mere definition of the centres of similitude or perspective of two circles, it is evident that: 1°. When the circles intersect, they connect with each point of intersection by the two bisectors, external and internal, of the angle between the radii, and therefore (23) of the angle between the circles at the point (Euc. VI. 3); 2°. When the circles touch, one of them, the external or the internal according to circumstances, coincides with the point of contact; 3°. When the circles are equal and not concentric, they bisect, externally and internally, the interval between the two centres; 4°. When the circles are concentric and not equal, they both coincide with the common centre; 5°. When the circles are at once concentric and equal, one, the internal, coincides with the common centre, while the other, the external, is entirely indeterminate (15); 6°. When one circle is a point and the other not, they both coincide with the point; 7°. When one circle is a line and the other not, they coincide with the extremities of the diameter of the latter whose direction is perpendicular to the former; 8°. When both circles are points, with the exception of dividing, externally and internally, in a common ratio the interval between the points, they are otherwise both indeterminate (13);

and 9°. When both circles are lines, they connect from infinity, as in 1°, with the point of intersection by the two bisectors, external and internal, of the angle determined by the lines. Of these particulars, some, less evident than the others, will appear more fully from the general properties of the centres of similitude or perspective of any two circles, which will form the main subject of the present chapter.

When two circles, whatever be their nature, are given in magnitude and position, their two centres of perspective, external and internal, being in fact the two centres of perspective, external and internal, of any pair of their parallel diameters (131), are of course implicitly given with them; and, as already stated in (44), possess with respect to the circles, considered as similar figures at once similarly and oppositely placed, all the properties of the corresponding centres of similitude of similar figures of any form similarly or oppositely placed; all lines passing through either intersecting them at equal angles, dividing them into pairs of similar segments, determining on them pairs of homologous points at which the radii and tangents are parallel, and intercepting in them pairs of homologous chords in the constant ratio of the radii; and the two particular lines, real or imaginary, which are tangents to either circle being tangents to the other also (42).

196. The circle on the interval between the centres of similitude of two circles as diameter, which when the circles intersect passes evidently through the two points of intersection (195, 1°), is sometimes called *the circle of similitude* of the circles, and may be easily shown to be always coaxal with them, and to be such that from every point of it they subtend equal angles, real or imaginary.

For, the distances of every point on it from their centres having, by (158), the constant ratio of their radii, therefore, by pairs of similar right-angled triangles, the tangents to them from every point on it have the same constant ratio; but because the ratio of the tangents to them from every point of it is constant, it is coaxal with them (192, Cor. 1°); and because the constant ratio is that of their radii, the pairs of tangents to them from every point of it contain equal angles, real or imaginary, and therefore &c.

Or, more briefly, thus: the tangents, real or imaginary, to two circles from each of their centres of similitude having the ratio of their radii (44), therefore, by (192, Cor. 1°), so have the tangents, real or imaginary, to them from every point of the circle of which the interval between the two centres of similitude is diameter, and therefore &c.

We shall see, in the next article, that the three circles of similitude of the three groups of two determined by any system of three arbitrary circles, besides being thus coaxal each with the two original circles of its own group, are also coaxal with each other.

197. *For any three circles, whose centres are A, B, C, and radii AR, BS, CT, if X and X', Y and Y', Z and Z' be the three pairs of centres of similitude, external and internal, of the three groups of two whose centres are B and C, C and A, A and B, respectively, then—*

1°. *The six points X and X', Y and Y', Z and Z' lie three and three on four lines.*

2°. *The six lines AX and AX', BY and BY', CZ and CZ' pass three and three through four points.*

3°. *The three middle points of the three segments XX', YY', ZZ' are collinear.*

4°. *The three circles of which the three segments XX', YY', ZZ' are diameters are coaxal.*

Of these properties, the two first follow at once from the general criteria a and b' of Art. 134, by virtue of the relations (195) which determine the three pairs of points X and X', Y and Y', Z and Z' on the three sides BC, CA, AB of the triangle ABC, viz.:

$$\frac{BX}{CX}=+\frac{BS}{CT},\quad \frac{CY}{AY}=+\frac{CT}{AR},\quad \frac{AZ}{BZ}=+\frac{AR}{BS},$$

$$\frac{BX'}{CX'}=-\frac{BS}{CT},\quad \frac{CY'}{AY'}=-\frac{CT}{AR},\quad \frac{AZ'}{BZ'}=-\frac{AR}{BS};$$

and the two last follow immediately from the first, by virtue of the two general properties 1° and 4° of Cor. 1°, Art. 189, of which they furnish obvious examples; or they may be established independently as follows.

If U, V, W be the three middle points of the three segments XX', YY', ZZ', then since, by (150),

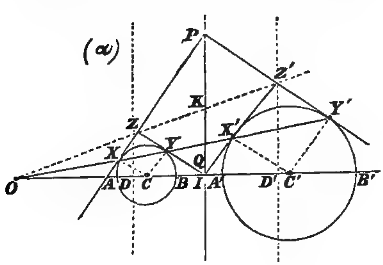
$$\frac{BU}{CU}=\frac{BS^2}{CT^2},\quad \frac{CV}{AV}=\frac{CT^2}{AR^2},\quad \frac{AW}{BW}=\frac{AR^2}{BS^2},$$

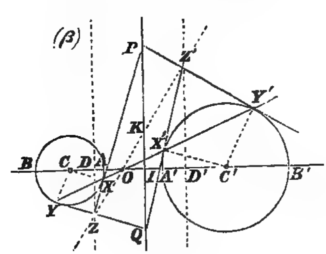
therefore, by (134, *a*), the three points U, V, W are collinear; and because the centres of the three circles of which the three segments XX', YY', ZZ' are diameters are collinear, the three circles themselves, being, by the preceding (196), coaxal each with the corresponding pair of the original circles to which it is the circle of similitude, are therefore, by (190, 3°), coaxal with each other.

The four lines $Y'Z'X$, $Z'X'Y$, $X'Y'Z$, and XYZ, on which the six points X and X', Y and Y', Z and Z', by property 1°, are grouped three and three, are termed, from their origin, the four *axes of similitude* of the three original circles, and occur very frequently in Modern Geometry in enquiries connected with systems of three circles. As passing each through a centre of similitude of every two of the three, they each, if they meet the three circles at all, intersect them at three equal angles; determine on them two systems of three points at which the radii and tangents are parallel; intercept in them three chords in the ratios of their radii; and, if happening to touch one of the three, touching the other two also (42).

The four axes of similitude of any system of three circles furnish evidently the four solutions of the problem "to draw a line intersecting the three circles at equal angles."

198. As every line passing through either centre of similitude O, external (fig. α) or internal (fig. β), of any two circles whose centres are C and C', meets them at two pairs of *homologous*





points (39) X and X', Y and Y', at which the two pairs of corresponding radii CX and $C'X'$, CY and $C'Y'$, and of corresponding tangents ZX and $Z'X'$, ZY and $Z'Y'$ are parallel (42); so it meets them at two pairs of *anti-homologous* points, as they are termed, X and Y', Y and X', at which the two pairs of corresponding radii CX and $C'Y'$, CY and $C'X'$, and of corresponding tangents ZX and $Z'Y'$, ZY and $Z'X'$, though not parallel, make equal angles and determine isosceles triangles with the line. Hence any two circles C and C' may be conceived to be divided by a variable line revolving round either of their centres of similitude O, and simultaneously exhausting them both, either into pairs of homologous points X and X' or Y and Y', or into pairs of anti-homologous points X and Y' or Y and X'; the distances of every two of the former from the centre of similitude to which they correspond having, as already shown in (42), a constant *ratio* termed that of *the similitude* of the figures, and the distances of every two of the latter from the same having, as may be easily shewn, a constant *product* termed that of *the anti-similitude* of the figures.

For, since, by (Euc. III. 35, 36), the two rectangles $OX.OY$ and $OX'.OY'$ are both constant, and since, by (42), the two ratios $OX : OX'$ and $OY : OY'$ are both constant and equal, therefore the two rectangles $OX.OY'$ and $OY.OX'$ are both constant and equal, and therefore &c.

The constant ratio $OX : OX'$ or $OY : OY'$ being positive or negative according as O is the external or the internal centre of similitude, and the two constant rectangles $OX.OY$ and $OX'.OY'$ being both positive and both negative together according as O is external or internal to the circles; hence, as regards the two centres of similitude of any two real circles, the constant rectangle of anti-similitude $OX.OY'$ or $OY.OX'$ is positive for the external and negative for the internal, positive for the internal and negative for the external, or, positive for both, according as the distance between the centres of the circles CC' is greater than the sum, less than the difference, or, intermediate between the sum and difference, of their radii.

199. From the properties of the preceding article, it follows evidently, conversely, that—

If on a variable line, revolving round a fixed point O and intersecting a fixed circle C in two variable points X and Y, two variable points Y' and X' be taken, such that $OX.OY' = OY.OX'$ = any constant magnitude, positive or negative; the locus of the two points Y' and X' is another circle C', with respect to which and the original the point O is a centre of similitude, the external or the internal according as the two constant rectangles $OX.OY'$, or $OY.OX'$, and $OX.OY$ have similar or opposite signs.

For, if on the diameter AB of the original circle which passes through O (figs. of last article) the two points B' and A' be taken for which $OA.OB' = OB.OA'$ = the given rectangle, the circle on $A'B'$ as diameter fulfils evidently, by the preceding, the conditions of the required locus; but since, as regards it and the original, if C' be its centre, as $OA.OB' = OB.OA'$, therefore

$$OA : OA' = OB : OB' = OC : OC' = CA : CA' = CB : C'B',$$

and therefore &c.

If D and D' (same figures) be the two inverses of the point O with respect to the two circles; since then, by (164),

$$OX.OY = OC.OD \text{ and } OX'.OY' = OC'.OD',$$

therefore the constant product of anti-similitude for the point O, viz.,

$$OX.OY' \text{ or } OY.OX' = OC.OD' \text{ or } OC'.OD;$$

a value found very useful in the modern Theory of Inversion.

200. The two products of anti-similitude, external and internal, for any two circles may be expressed, in terms of their radii and of the distance between their centres, as follows:

If (same figures as before) C and C' be their two centres, CR and $C'R'$ their two radii, and O either centre of similitude, external (fig. α) or internal (fig. β), then since (Euc. III. 35, 36),

$$OX.OY = OC^2 - CR^2 \text{ and } OX'.OY' = OC'^2 - C'R'^2,$$

and since (42)

$$OX : OX' = OY : OY' = OC : OC' = \pm (CR : C'R'),$$

according as O is external or internal, therefore $OX.OY'$, or its equivalent $OY.OX'$, $= OC.OC' \mp CR.C'R'$; but, by (84),

$$OC = \frac{CR}{CR \mp C'R'}.C'C \text{ and } OC' = \frac{C'R'}{C'R' \mp CR}.CC',$$

therefore, denoting by r and r' the two radii and by d the distance between the two centres, the two products of anti-similitude, external and internal, have respectively for values

$$\frac{rr'}{(r-r')^2}\cdot\{d^2-(r-r')^2\} \text{ and } \frac{rr'}{(r+r')^2}\cdot\{(r+r')^2-d^2\};$$

which are the formulæ by which to calculate them in numbers when the centres and the radii of the circles are given, and which for real circles, it will be observed, give them signs in exact accordance with the particulars already stated in Art. 198.

201. The two circles round the two centres of similitude of any two circles as centres, the squares of whose radii are equal in magnitude and sign to the corresponding rectangles of anti-similitude, are termed the two *circles of anti-similitude*, external and internal, of the original circles. When the latter intersect, they evidently (198) pass through their two points, and bisect, externally and internally, their two angles, of intersection, and are therefore in that case coaxal with them and with their circle of similitude (196); that they are so in all cases may easily be shewn as follows:

Since for each centre of similitude O (same figures as before)

$$OX : OX' = OY : OY' = OC : OC',$$

therefore

$$OC'.OX.OY - OC.OX'.OY'$$
$$= (OC - OC').(OX.OY' \text{ or } OY.OX'),$$

but $OC - OC' = C'C$, and $OX.OY'$ or its equivalent $OY.OX'$ = the square of the radius of the circle of anti-similitude round O, $= -OX''.OY''$, if X'' and Y'' be any two diametrically opposite points of that circle; therefore for the three circles whose collinear centres are C, C' and O,

$$OC'.OX.OY - OC.OX'.OY' = CC'.OX''.OY'',$$

and therefore by (192, Cor. 6°) those three circles are coaxal.

As every two anti-homologous points with respect to either centre of similitude of two circles are evidently inverse points (149) with respect to the circle of anti-similitude corresponding to that centre, it follows therefore, from (156), that *every circle passing through any pair of anti-homologous points with respect to*

either centre of similitude of two circles intersect at right angles the circle of anti-similitude corresponding to that centre.

Again, as every circle orthogonal to two others is orthogonal to every circle coaxal with the two (187, 4°), it follows, of course, from the relations of coaxality, established above and in (196), between any two circles, their circle of similitude, and their two circles of anti-similitude, that *every circle orthogonal to two others is orthogonal at once to their circle of similitude and also to their two circles of anti-similitude.*

202. *For any three circles A, B, C, if D and D', E and E', F and F' be the three pairs of circles of anti-similitude, external and internal, of B and C, C and A, A and B respectively; then—*

1°. *The four groups of three circles E', F', D; F', D', E; D', E', F; and D, E, F are coaxal.*

2°. *Their four radical axes pass through the radical centre of the original group A, B, C.*

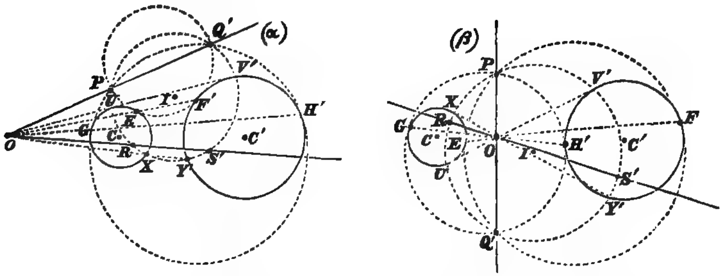
3°. *Their four pairs of common points, real or imaginary, are inverse pairs with respect to the four axes of similitude, and to the orthogonal circle, of the group A, B, C.*

4°. *Their four pairs of limiting points, imaginary or real, are the intersections of the four axes of similitude with the orthogonal circle of the group A, B, C.*

These several properties follow immediately from, or rather are all examples of, the general properties of Art. 190; the three pairs of circles D and D', E and E', F and F' being coaxal with the three pairs B and C, C and A, A and B (201); the four groups of centres of the four groups of circles E', F', D; F', D', E; D', E', F; and D, E, F being collinear (197, 1°); their four lines of centres Y', Z', X; Z', X', Y; X', Y', Z; and X, Y, Z being the four axes of similitude of the group A, B, C (197); and the whole six circles themselves being all cut orthogonally by the common circle, real or imaginary, orthogonal to the three A, B, C (183, Cor. 1°).

203. As for any two circles, regarded as similar figures, every two points P and P', or Q and Q', whether on the circles or not, which connect through either centre of similitude O, and which are such that the ratio of their distances from it $OP : OP'$, or $OQ : OQ'$, is equal in magnitude and sign to the constant ratio

of similitude for it, are termed homologous points with respect to it (42); so for any two circles, regarded as anti-similar figures, every two points P and Q', or R and S', whether on the



circles or not, which connect through either centre of anti-similitude O, external (fig. α) or internal (fig. β), and which are such that the product of their distances from it $OP.OQ'$, or $OR.OS'$, is equal in magnitude and sign to the constant product of anti-similitude for it, are termed anti-homologous points with respect to it. And again, as in the former case, every two connectors PQ and $P'Q'$ of two points P and Q, and of their two homologues P' and Q' with respect to either centre of similitude O, are termed homologous lines with respect to that centre (39, 6°); so, in the latter case, every two connectors PR and $Q'S'$ of two points P and R, and of their two anti-homologues Q' and S' with respect to either centre of anti-similitude O, external (fig. α) or internal (fig. β), are termed anti-homologous lines with respect to that centre.

It is evident (Euc. III. 35, 36) that every two pairs of anti-homologous points P and Q', R and S' with respect to either centre of similitude O of two circles, whether on the circles or not, lie, when not collinear, on a circle, the square of the tangent to which from that centre is equal in magnitude and sign to the corresponding product of anti-similitude of the circles; and, conversely, that every circle passing through any pair of anti-homologous points P and Q' with respect to either centre of similitude O of two circles, whether on the circles or not, determines pairs of anti-homologous points R and S', real or imaginary, with respect to that centre on all lines passing through it; intersects the circles themselves in two pairs of anti-homologous

points U and V', X and Y', real or imaginary, with respect to the same; and, when, by the coincidence of the two points of intersection at E or G, touching either circle, then, by the simultaneous coincidence of the two anti-homologous points of intersection at F' or H', touching the other also (19).

It is evident also that, in their more general as in their more restricted acceptation (198), all pairs of anti-homologous points P and Q', R and S', &c. with respect to either centre of similitude O of two circles are inverse pairs with respect to the circle of anti-similitude corresponding to that centre (201); and that, consequently, all circles passing, as in the above figures, through any pair of them P and Q', with respect to either centre O, intersect at right angles the circle of anti-similitude corresponding to that centre (156).

204. *All pairs of homologous tangents with respect to either centre of similitude of two circles intersect on the line at infinity.*

All pairs of anti-homologous tangents with respect to either centre of similitude of two circles intersect on their radical axis.

For, if X and X' or Y and Y' (figures of Art. 198) be any pair of homologous points on the circles, X and Y' or Y and X' any pair of anti-homologous points on the same, and O the centre of similitude, external or internal, to which they correspond; then the two pairs of tangents at the former being parallel (41) intersect therefore on the line at infinity (16); and the two pairs at the latter determining isosceles triangles XPY' and YQX' with the line of the points (198) intersect therefore on the radical axis (182, Cor. 1°).

Conversely, *if from any point either on the line at infinity or on the radical axis of two circles four tangents be drawn to the circles, their four chords of contact with different circles intersect two and two at the two centres of similitude, external and internal, of the circles.*

For, the four tangents being parallel in the case of the line at infinity (16) and equal in the case of the radical axis (182, Cor. 1°), their four chords of contact with different circles in either case make equal angles with the circles, and therefore &c. (42).

Cor. Since, in the converse property, the two chords of contact of the two pairs of tangents to the same circles in-

tersect, in either case, on the line containing the point, and pass, in either case, through its two poles with respect to the two circles, see Art. 182, Cor. 10°; it follows consequently, from that property, that—

The two centres of perspective, external and internal, of any two chords of two circles which pass through the two poles with respect to the circles either of the line at infinity or of their radical axis, and which intersect on the line whichever it be, are the two centres of similitude, external and internal, of the circles.

205. *The interval between the polars of either centre of similitude of two circles, of course bisected externally by the line at infinity, is bisected internally by the radical axis of the circles.*

For, if X and Y, X' and Y' (same figures as before) be the four intersections with the circles of any line passing through either centre of similitude O; then since, of the four vertices of the parallelogram $PZQZ'$ determined by their four tangents (41), the two opposites Z and Z', at which the pairs of tangents to the same circles intersect, lie on the two polars of O with respect to the two circles (166, Cor. 3°), and the two opposites P and Q, at which the pairs of tangents to different circles intersect, lie on the radical axis of the circles (204); and since in every parallelogram the two diagonals mutually bisect internally (Euc. I. 34), therefore &c.

Conversely, *if the interval between two homologous points with respect to either centre of similitude of two circles, of course bisected externally by the line at infinity, be bisected internally by the radical axis of the circles, the two points lie on the two polars of that centre of similitude with respect to the circles.*

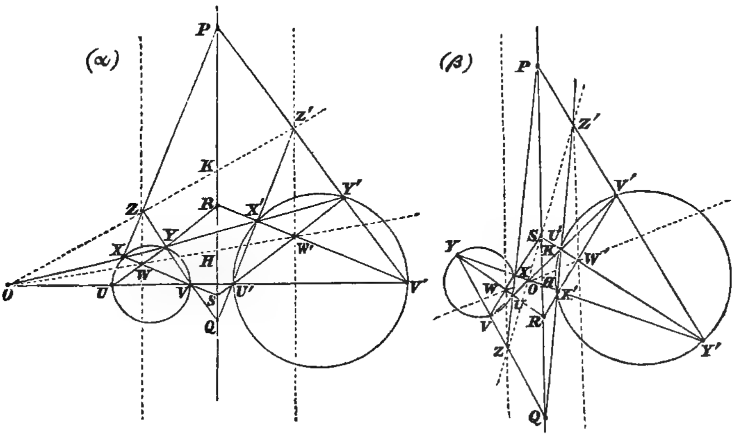
For, if connected with another pair of homologous points on the two polars in question, the interval between which, by the above, is also bisected by the radical axis, the two homologous connectors, being parallel to each other (41), would be parallel to the radical axis (Euc. VI. 2), and therefore &c.

Cor. It follows evidently from the above, as proved before for a particular case in (182, Cor. 9), that *for any two circles, however circumstanced as to magnitude and position, the two polars of the two centres of similitude with respect to each are equidistant in the two.*

206. The two properties of Art. 204 are evidently particular cases of the two following, viz.—

All pairs of homologous chords with respect to either centre of similitude of two circles intersect on the line at infinity.

All pairs of anti-homologous chords with respect to either centre of similitude of two circles intersect on their radical axis.



For, if UX and $U'X'$, or UY and $U'Y'$, or VX and $V'X'$, or VY and $V'Y'$ be any pair of homologous chords of the circles, UX and $V'Y'$, or UY and $V'X'$, or VX and $U'Y'$, or VY and $U'X'$ any pair of anti-homologous chords of the same, and O the centre of similitude, external (fig. α) or internal (fig. β), to which they correspond; then since, by the similitude of the figures,

$$OU : OU' = OV : OV' = OX : OX' = OY : OY',$$

therefore the directions of the several pairs of homologous chords are parallel, and therefore the several pairs themselves intersect on the line at infinity (16); and since, by the anti-similitude of the figures,

$$OU.OV' = OV.OU' = OX.OY' = OY.OX',$$

therefore the extremities of the several pairs of anti-homologous chords are concyclic, and therefore the several pairs themselves intersect on the radical axis of the circles (182, Cor. 3°).

In the two parallelograms $PZQZ'$ and $RWSW'$ (see figs.) formed by the four pairs of homologous and of anti-homologous

chords determined by any two lines passing through O and intersecting the circles, the two diagonals ZZ' and WW', which connect the homologous intersections of pairs of chords of the same circles, being both bisected by the radical axis, their extremities lie consequently, as in the preceding (205), on the two polars of the point O with respect to the two circles; lines which with respect to that point possess evidently the property peculiar to themselves of being at once homologous and anti-homologous chords of the figures.

In the application of the above properties to any system of two circles, it is evident, from Art. 204, Cor., that—

All pairs of chords passing through the two poles of and intersecting upon the line at infinity are homologous pairs with respect to both centres of similitude.

All pairs of chords passing through the two poles of and intersecting upon the radical axis are anti-homologous pairs with respect to both centres of similitude.

207. The two general properties of the preceding article establish, as stated in (144), the quadruple relation of perspective existing between every two circles in the same plane, however circumstanced as to position and magnitude; the first their double relation of perspective as similar figures at once similarly and oppositely placed, and the second their double relation of perspective as anti-similar figures at once similarly and oppositely placed; the line at infinity and their radical axis being the axes of their double perspective in the two cases respectively, and the two centres of similitude or of anti-similitude, external and internal, being the centres of their double perspective in both cases alike.

As every two figures in perspective, whatever be their nature (141), evidently intersect their axis of perspective, whatever be its position, (or each axis of perspective if like two circles they have more than one), at the same system of points, real or imaginary, whose number depends, of course, on the nature of the figures; it follows, consequently, from the above, that *for every two circles in the same plane, however circumstanced as to magnitude and position, the radical axis and the line at infinity, being both axes of perspective, are both chords of intersection; the corresponding points of intersection, real or imaginary, according*

to circumstances in the case of the former, being of course from the nature of the figures always imaginary in the case of the latter. This remarkable conclusion, as regards the line at infinity in relation to every two circles, the reader will find abundantly verified by various other considerations in the course of the sequel.

As again, every two figures in perspective, whatever be their nature, subtend, as stated in (41), their centre of perspective, whatever be its position, (or each centre of perspective if like two circles they have more than one), in the same system of tangents, real or imaginary, whose number depends, as before, on the nature of the figures. Hence, and from the above, the following pair of analogous properties respecting the two centres and the two axes of perspective of every two circles in the same plane, viz.—

Every two circles in the same plane, however circumstanced as to magnitude and position, subtend the same two angles, real or imaginary, at their two centres of perspective.

Every two circles in the same plane, however circumstanced as to magnitude and position, intercept the same two segments, real or imaginary, on their two axes of perspective.

208. The following pairs of polar relations, common respectively to both centres and to both axes of perspective of two circles, supply additional illustrations of the analogy noticed at the close of the preceding article, viz.—

a. The two poles of every line through either centre of perspective of two circles connect through the same centre of perspective.

a'. The two polars of every point on either axis of perspective of two circles intersect on the same axis of perspective.

For, in the former case, the two polars of the line are evidently homologous points with respect to the centre, whichever it be, and therefore &c. (41); and, in the latter case, the property is evidently that already noticed in Art. 204, Cor., and therefore &c.

b. Every two lines through either centre of perspective of two circles which are conjugates with respect to either circle are conjugates with respect to the other also.

b'. Every two points on either axis of perspective of two circles

which are conjugates with respect to either circle are conjugates with respect to the other also.

For, in the former case, the two lines, passing each through the pole of the other with respect to one of the circles (174), pass, therefore, by (a), each through the pole of the other with respect to the other circle also, and therefore &c.; and, in the latter case, the two points, lying each on the polar of the other with respect to one of the circles (174), lie, therefore, by (a'), each on the polar of the other with respect to the other circle also, and therefore &c.

c. In every two circles the two centres of perspective are those of every two inscribed chords whose poles coincide on either axis of perspective.

c'. In every two circles the two axes of perspective are those of every two circumscribed angles whose polars coincide through either centre of perspective.

For, in the former case, the four extremities of the two chords determine, according to the axis, two pairs either of homologous or of anti-homologous points with respect to both centres of perspective (204), and therefore &c.; and, in the latter case, the four sides of the two angles determine, whichever be the centre, two pairs of homologous and two pairs of anti-homologous tangents with respect to the centre (204), and therefore &c.

d. In every two circles the two centres of perspective divide, externally and internally, in common ratios the intervals between the two poles of each axis of perspective.

d'. In every two circles the two axes of perspective divide, externally and internally, in common ratios the intervals between the two polars of each centre of perspective.

For, in the latter case, the two axes of perspective, as already shewn in (205), bisect, externally and internally, the intervals between the polars of each centre of perspective, and therefore &c.; and, in the former case, the two centres of perspective being, by (204, Cor.), those of every pair of chords of the circles which pass through the poles of and intersect on either axis of perspective, are therefore those of the particular pair perpendicular to the line of centres, the interval between which they consequently divide, externally and internally, in the ratio of their lengths, and therefore &c.

e. When two circles intersect at right angles, the polar of either centre of perspective with respect to either circle is the polar of the other centre of perspective with respect to the other circle.

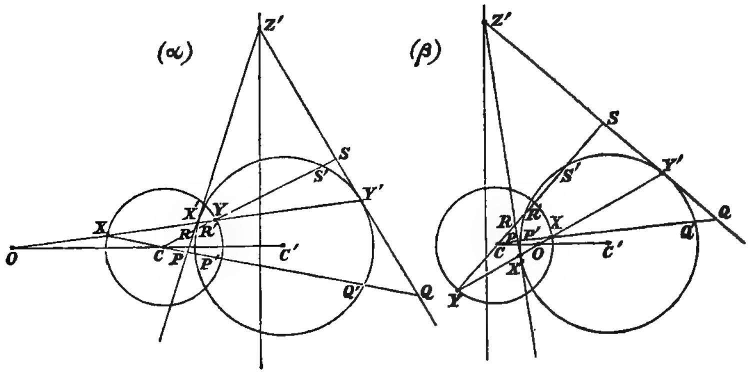
e′. When two circles intersect at right angles, the pole of either axis of perspective with respect to either circle is the pole of the other axis of perspective with respect to the other circle.

For, in the latter case, the centre of each circle being the pole of the line at infinity with respect to itself (165, 3°), and the pole of the radical axis with respect to the other (165, 6°), therefore &c.; and, in the former case, as the two lines connecting either point of intersection of the circles with their two centres of perspective make each half a right angle with each of the two radii at the point of intersection (195, 1°), therefore the two lines from either point of intersection which make each half a right angle with the line of centres of the circles intersect that line at two points, each of which, by (158, Cor. 1°), is the inverse of one centre of perspective with respect to one circle, and of the other centre of perspective with respect to the other circle, and therefore &c. (165).

f. When two circles intersect at right angles, every two tangents to either which intersect on a polar of either centre of perspective are conjugate lines with respect to the other.

f′. When two circles intersect at right angles, every two points of either which connect through a pole of either axis of perspective are conjugate points with respect to the other.

For, in the latter case, the centres of the two circles being the poles of both their axes of perspective (165, 3°, 6°), and the extremities of all diameters of either being conjugate points with respect to the other (177), therefore &c.; and, in the former case, if C and C' be the centres of the two circles, O either of their centres of perspective, the external (fig. α) or the internal (fig. β), X and Y, X' and Y' their two pairs of intersections with any line passing through O, $X'Z'$ and $Y'Z'$ the two tangents to either C' at its pair of intersections X' and Y', which, by (166, Cor. 3°), intersect on the polar of O with respect to itself, P and Q, R and S their two pairs of intersections with the two homologous radii CX and CY of the other C, to which, by (198, and Euc. III. 18), they are respectively at right angles, and, P' and Q', R' and S' the two pairs of intersections, real or



imaginary, of their circle with the same radii; then, since, by the isosceles triangle $X'Z'Y'$, the two angles at X' and Y' are equal, therefore, by (63), or by (134, a),

$$PX'^2 : QY'^2 = PX^2 : QX^2, \text{ and } RX'^2 : SY'^2 = RY^2 : SY^2,$$

and therefore, by Euc. III. 35, 36,

$$PP'.PQ' : QP'.QQ' = PX^2 : QX^2,$$

and $$RR'.RS' : SR'.SS' = RY^2 : SY^2;$$

but, since the circles, by hypothesis, intersect at right angles, therefore, by (156),

$$CP'.CQ' = CX^2, \text{ and } CR'.CS' = CY^2,$$

and therefore, by (161, Cor. 1°),

$$CP.CQ = CX^2, \text{ and } CR.CS = CY^2,$$

from which, since, by (198), the two tangents $Z'X'$ and $Z'Y'$ are perpendiculars to the two radii CX and CY, it follows, consequently, from (165), that they are the polars of the two points Q and R with respect to the circle C, and therefore &c. (174).

209. *Every circle having contacts of similar species with two others touches them at a pair of anti-homologous points with respect to their external centre of perspective.*

Every circle having contacts of opposite species with two others touches them at a pair of anti-homologous points with respect to their internal centre of perspective.

For, if C and C' (figures, Art. 198) be the centres of the two touched circles, and X and Y', or Y and X', their two

points of contact with the touching circle; then since the chord of contact XY', or YX', makes equal angles with the radii of the latter, it does so with those of the former at its extremities, and therefore (42) passes through a centre at perspective of the former, the external (fig. α) or the internal (fig. β), according as their radii CX and $C'Y'$, or CY and $C'X'$, at its extremities are at similar or opposite sides of it (44); that is, according as the contacts of the touching with the touched circles are of similar or opposite species, and therefore &c.

Conversely, *Every circle passing through a pair of anti-homologous points with respect to either centre of perspective of two others, and touching either with contact of either species, touches the other with contact of similar or opposite species, according as the centre of perspective is external or internal.*

For the line XY', or YX', (same figures as before), passing through a pair of anti-homologous points X and Y', or Y and X', with respect to a centre of perspective O of the two circles whose centres are C and C', makes equal angles with their radii at the points, and also with those of every circle passing through the points; consequently, if the latter circle have contact of either species with either of the former, it has contact of similar or opposite species with the other, according as their radii at the points, (those of itself lying necessarily at the same side), lie at similar or opposite sides of the line; that is, according as the centre of perspective is external (fig. α) or internal (fig. β), and therefore &c. See also Art. 203.

Cor. 1°. Every two anti-homologous points with respect to either centre of perspective of two circles being inverse points with respect to the corresponding circle of anti-similitude (203), it follows at once (156) from the first part of the above, that—

Every circle having contacts of similar species with two others intersects at right angles their external circle of anti-similitude.

Every circle having contacts of opposite species with two others intersects at right angles their internal circle of anti-similitude.

Properties which, as the two circles of anti-similitude are coaxal with the original circles, coincide consequently with those already established on other principles in (193, Cor. 2°), viz. that—

Every circle having contacts of similar species with two others intersects at right angles the coaxal circle whose centre is their external centre of perspective.

Every circle having contacts of opposite species with two others intersects at right angles the coaxal circle whose centre is their internal centre of perspective.

Cor. 2°. Since when a number of circles are orthogonal to the same circle, the radical axis of every two of them passes through, and the radical centre of every three of them coincides with, its centre; it follows consequently, from Cor. 1°, or indeed again directly from the first part of the above, that—

For every two circles having contacts of similar species with two others, the radical axis passes through their external centre of perspective.

For every two circles having contacts of opposite species with two others, the radical axis passes through their internal centre of perspective.

For every three circles having contacts of similar species with two others, the radical centre coincides with their external centre of perspective.

For every three circles having contacts of opposite species with two others, the radical centre coincides with their internal centre of perspective.

Cor. 3. Since when three circles are orthogonal to three others, both systems are coaxal and conjugate to each other (185), it follows also from Cor. 1°, or again, directly from the first part of the above, that—

The circle orthogonal to three others, and the two circles touching the three with contacts of similar species, are coaxal, and have for radical axis the line passing through the three external centres of perspective of the three groups of two contained in the three (197, 1°).

The circle orthogonal to three others, and the two circles touching the same two of them with contacts of similar species and the third with contact of the opposite species, are coaxal, and have for radical axis the line passing through the external centre of perspective of the two and the two internal centres of perspective of the two combined each with the third (197, 1°).

COR. 4°. The second part of the above supplies obvious and rapid solutions of two following problems, viz.—

To describe a circle passing through a given point and having contacts of similar or opposite species with two given circles.

For by it, (see figures Art. 203), the two circles, real or imaginary, passing through the given point P and its antihomologue Q' with respect to either centre of perspective O of the given circles, and touching either circle, touch the other with contact of similar or opposite species, according as O is (fig. α) the external or (fig. β) the internal centre of perspective of the two, and therefore &c.

Of the four circles supplied in pairs by the two cases of the above, each evidently is the unique solution of some one of the four different cases of the more definite problem: "*To describe a circle passing through a given point and having contacts of assigned species with two given circles.*"

COR. 5°. Since if a circle O have contacts of definite species with three given circles A, B, C, a concentric circle O' passing through the centre of any one of them C evidently touches with contacts of definite species two circles A' and B' concentric with the other two A and B, whose radii are equal to the sums or differences, according to circumstances, of the radii of A and C and of B and C, and which are therefore given with the latter; hence the unique solution of the definite problem: "*To describe a circle having contacts of given species with three given circles,*" is reduced at once to that of the definite problem just stated: "*To describe a circle passing through a given point and having contacts of given species with two given circles;*" and, consequently, the eight different solutions of the celebrated problem: "*To describe a circle touching three given circles,*" corresponding to the eight different combinations of contacts of both kinds with the three, may be regarded as all given in detail by so many applications of the definite construction of Cor. 4°, which, though indirect, is perhaps on the whole the simplest of which they are susceptible, see 183, Cor. 6° and 186, 3°.

Of all the constructions ever given for the direct determination of the eight circles of contact of three given circles, that of M. Gergonne, who regarded them as divided into four con-

jugate pairs having contacts of opposite species with the three given circles, and who determined simultaneously the six points of contact of each pair of conjugates, is decidedly the most elegant. The principles on which it depends are contained also in the above, and form the subject of the next article.

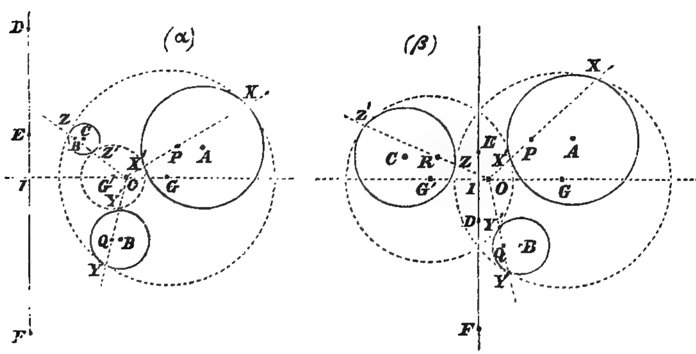
210. *When two circles have contacts of opposite species with each of three others.*

a. If they have each contacts of similar species with the three, their radical axis passes through the three external centres of perspective of the three groups of two contained in the three.

b. If they have each contacts of similar species with the same two of the three and contact of the opposite species with the third, their radical axis passes through the external centre of perspective of the two and through the two internal centres of perspective of the two combined each with the third.

c. Their three chords of contact with the three pass, in either case, through the radical centre of the three and through the three poles of their radical axis with respect to the three.

For, if A, B, C be the three centres of the touched circles,



X, Y, Z and X', Y', Z' their six points of contact with the two touching each with contacts of opposite species and having themselves each either contacts of similar species with all three (fig. α), or contacts of similar species with two of them A and B and contact of the opposite species with the third C (fig. β), D, E, F the three centres of perspective, external or internal, of B and C, C and A, A and B respectively, through which

the three pairs of connectors YZ and $Y'Z'$, ZX and $Z'X'$, XY and $X'Y'$ by (209) pass, and O the internal centre of perspective of XYZ and $X'Y'Z'$, through which the three connectors XX', YY', ZZ' by the same pass; then, since, by (198),

$$DY.DZ = DY'.DZ', \quad EZ.EX = EZ'.EX', \quad FX.FY = FX'.FY',$$

therefore, by (182, Cor. 1°), the line DEF (197, 1°) is the radical axis of the two circles XYZ and $X'Y'Z'$, which proves a and b; since, by (198), $OX.OX' = OY.OY' = OZ.OZ'$, therefore, by (183, Cor. 1°), the point O is the radical centre of the three circles A, B, C, which proves the first part of c; and, since, by (182, Cor. 1°), the three pairs of tangents at X and X', Y and Y', Z and Z' to the two circles XYZ and $X'Y'Z'$ intersect on their radical axis DEF, therefore, by (166, Cor. 3°), their three chords of contact XX', YY', ZZ' pass through the three poles P, Q, R of that line with respect to the three circles A, B, C, which proves the second part of c; and therefore &c.

Cor. 1°. Hence the following elegant construction of M. Gergonne for determining directly the six points of contact X, Y, Z and X', Y', Z' of any particular conjugate pair of the eight circles of contact of three given circles A, B, C.

Take the axis of similitude DEF of the three given circles (197) which, by the above (a or b), is the radical axis of the conjugate pair whose points of contact are required, and connect its three poles P, Q, R with respect to the given circles with their radical centre O; the three connecting lines OP, OQ, OR intersect the three circles in three pairs of points, real or imaginary, X and X', Y and Y', Z and Z', which, by the above (c), are the six points required.

The unique solution of the definite problem, "to describe a circle having contacts of given species with three given circles," is of course involved in this construction, which with its three points of contact gives evidently those of its conjugate at the same time.

Cor. 2°. If G and G' be the centres of the two circles XYZ and $X'Y'Z'$, since, by (199) and (181), the line GG' passes through the point O and intersects at right angles the line DEF. Hence—

For the eight circles of contact of any system of three circles, the eight centres lie two and two in conjugate pairs on the four perpendiculars to the four axes of similitude through the radical centre of the three.

COR. 3°. The circle round O as centre, the square of whose radius is equal in magnitude and sign to the common value of the three equal rectangles $OX.OX'$, $OY.OY'$, $OZ.OZ'$, being, by (201), the internal circle of anti-similitude of the two XYZ and $X'Y'Z'$, and, by (183, Cor. 1°), the orthogonal circle of the three A, B, C. Hence—

Of the eight circles of contact of any system of three circles, the four conjugate pairs have a common internal centre and a common internal circle of anti-similitude, viz. the radical centre and the orthogonal circle of the three.

COR. 4°. Each circle of anti-similitude, external and internal, of any two circles being coaxal with the two (201). Hence by Cor. 3°.—

Of the eight circles of contact of any system of three circles, the four conjugate pairs belong to the four coaxal systems determined by the four axes of similitude with the orthogonal circle of the three. See Cor. 3°, Art. 209.

211. The two properties of Art. 209 are evidently particular cases of the two following, viz.—

For any system of two circles, every circle passing through any pair of anti-homologous points with respect to their external centre of perspective intersects them at equal angles, and every circle passing through any pair of anti-homologous points with respect to their internal centre of perspective intersects them at supplemental angles.

For, if C and C' (figs. Art. 203) be the centres of the two circles, P and Q' any pair of anti-homologous points with respect to either of their centres of perspective, the external (fig. α) or the internal (fig. β), UX and $V'Y'$ their pair of anti-homologous chords of intersection (203) with any circle passing through P and Q', and I the centre of that circle; then, since (Euc. I. 5) the two pairs of angles IUV' and $IV'U$, IXY' and $IY'X$ are equal, and since (198) the two pairs of angles CUV' and $C'V'U$, CXY' and $C'Y'X$ are equal (fig. α) or supplemental (fig. β),

therefore the two pairs of angles IUC and $IV'C'$, IXC and $IY'C'$ are equal (fig. α) or supplemental (fig. β), and therefore &c. (23).

Conversely, *For any system of two circles, every circle intersecting them at equal angles intersects them in a pair of anti-homologous chords with respect to their external centre of perspective, and every circle intersecting them at supplemental angles intersects them in a pair of anti-homologous chords with respect to their internal centre of perspective.*

For, if C and C' (same figures as before) be the centres of the two circles, UX and $V'Y'$ their two chords of intersection with any circle intersecting them at equal angles (fig. α) or at supplemental angles (fig. β), and I the centre of that circle; then, since (Euc. I. 5) the two pairs of angles IUV' and $IV'U$, IXY' and $IY'X$ are equal, and since, by hypothesis, the two pairs of angles IUC and $IV'C'$, IXC and $IY'C'$ are equal (fig. α) or supplemental (fig. β), therefore the two pairs of angles CUV' and $C'V'U$, CXY' and $C'Y'X$ are equal (fig. α) or supplemental (fig. β), and therefore &c. (198.)

Cor. 1°. Every two anti-homologous points with respect to either centre of perspective of two circles being inverse points with respect to the corresponding circle of anti-similitude (201), it follows at once from the second part of the above, precisely as in Cor. 1°, Art. 209, that—

Every circle intersecting two others at equal angles intersects at right angles their external circle of anti-similitude, and every circle intersecting two others at supplemental angles intersects at right angles their internal circle of anti-similitude.

Properties which, as both circles of anti-similitude are coaxal with the original circles, coincide evidently, as in the corollary referred to, with those already established on other principles in (193, Cor. 2°), viz. that—

Every circle intersecting two others at equal angles intersects at right angles the coaxal circle whose centre is their external centre of perspective, and every circle intersecting two others at supplemental angles intersects at right angles the coaxal circle whose centre is their internal centre of perspective.

Cor. 2°. Again, as in Cor. 2°, Art. 209, since, when a

number of circles are orthogonal to the same circle, the radical axis of every two of them passes through, and the radical centre of every three of them coincides with, its centre; it follows therefore, from Cor. 1°, as in the corollary referred to, that—

When two circles intersect two others at equal angles their radical axis passes through the external centre of perspective of the two, and when two circles intersect two others at supplemental angles their radical axis passes through the internal centre of perspective of the two.

When three circles intersect two others at equal angles their radical centre coincides with the external centre of perspective of the two, and when three circles intersect two others at supplemental angles their radical centre coincides with the internal centre of perspective of the two.

Cor. 3°. Again, as in Cor. 3°, Art. 209, since when three circles are orthogonal to three others, both systems are coaxal and conjugate to each other (185); it follows also from Cor. 1°, as in the corollary referred to, that—

Every three circles intersecting three others at equal angles are coaxal, and have for radical axis the line passing through the three external centres of perspective of the three groups of two contained in the three (197, 1°).

Every three circles intersecting the same two of three others at equal angles and the third at the supplemental angle are coaxal, and have for radical axis the line passing through the external centre of perspective of the two and the two internal centres of perspective of the two combined each with the third (197, 1°).

Cor. 4°. As the unique circle, real or imaginary, orthogonal to three others intersects the three at equal angles, and, at the same time, every two of the three at equal angles and the third at the supplemental angle, it follows immediately as a particular case of Cor. 3°, that—

The unique circle orthogonal to three others is coaxal with every two circles intersecting the three at equal angles, and also with every two intersecting the same two of them at equal angles and the third at the supplemental angle; the axis of similitude of the three external to them all in the former case, and that

external to the two and internal to the third in the latter case, being the corresponding radical axis of the system.

Cor. 5°. In the particular case where one of the two intersecting circles has one combination of the angle of intersection and its supplement, and the other the opposite combination of the same angle of intersection and its supplement, with the three; then, by the second part of Cor. 1°, for the same reason as in (210, Cor. 3°), the radical centre and orthogonal circle of the three are the internal centre and circle of anti-similitude of the two. Hence the following extension of the property Cor. 3°, of the preceding article, viz.—

The unique circle orthogonal to three others is the common internal circle of anti-similitude of every pair of conjugate circles intersecting the three at any opposite combinations of the same angle and its supplement.

Cor. 6°. The following properties of a variable circle intersecting a system of two or three fixed circles at equal or supplemental angles are evident, from Cors. 1°, 3° and 4° of the above, viz.—

a. A variable circle passing through a fixed point and intersecting two fixed circles at equal or at supplemental angles passes through a second fixed point, the anti-homologue of the first with respect to the corresponding centre of perspective of the circles.

b. A variable circle intersecting three fixed circles at equal or at any invariable combination of equal and supplemental angles describes the coaxal system determined by the corresponding axis of similitude with the orthogonal circle of the three.

Properties, the converses of which supply obvious solutions of the several problems of the three following groups, viz.—

To describe a circle (*a*) *passing through two given points and intersecting two given circles at equal or at supplemental angles,* (*b*) *passing through a given point and intersecting three given circles at equal or at any assigned combinations of equal and supplemental angles,* (*c*) *intersecting four given circles at equal or at any assigned combinations of equal and supplemental angles.*

212. With the two following properties of a system of three arbitrary circles, we shall conclude the present chapter and volume.

1°. *For any system of three circles, the three pairs of points, at which they are touched by the three pairs of circles tangent to one and orthogonal to the other two, lie on three circles, coaxal each with the two of the original three to which it does not correspond, and coaxal with each other.*

2°. *For any system of three circles, the three pairs of points, at which they are touched by any conjugate pair of their eight circles of contact, lie on three circles, coaxal each with the two of the original three to which it does not correspond, and coaxal with each other.*

For, if A_0, B_0, C_0 be the three circles, A, B, C their three centres, O their radical centre, P and P', Q and Q', R and R' the three pairs of points of contact in either case, and X_0, Y_0, Z_0 the three circles passing through them and having their three centres X, Y, Z on the three lines BC, CA, AB respectively; then, since P and P', Q and Q', R and R', in the case of 1°, by 186, 2°, are the intersections with A_0, B_0, C_0 of the three circles orthogonal to themselves and coaxal with B_0 and C_0, C_0 and A_0, A_0 and B_0 respectively, and in the case of 2°, by 210, *c.*, are collinearly distant from O by intervals such that the three rectangles $OP.OP'$, $OQ.OQ'$, $OR.OR'$ are equal in magnitude and sign, the first parts of both properties are evident; and it remains only to shew that in both cases the three points X, Y, Z on the three lines BC, CA, AB are collinear, in order to shew that in both cases the three circles X_0, Y_0, Z_0, of which they are the centres, are coaxal. See 190, 3°.

In the case of 1°, if α, β, γ be the three angles of intersection, real or imaginary, of the three pairs of original circles B_0 and C_0, C_0 and A_0, A_0 and B_0 respectively; then, since by the above, the three circles X_0, Y_0, Z_0 are coaxal with B_0 and C_0, C_0 and A_0, A_0 and B_0, and orthogonal to A_0, B_0, C_0 respectively, therefore, by 193, Cor. 1°,

$$\frac{BX}{CX}=\frac{BQ.\cos\gamma}{CR.\cos\beta},\quad \frac{CY}{AY}=\frac{CR.\cos\alpha}{AP.\cos\gamma},\quad \frac{AZ}{BZ}=\frac{AP.\cos\beta}{BQ.\cos\alpha},$$

and therefore &c. (134, *a.*). See also 193, Cor. 10°, where it was shewn, in a manner exactly similar, that three circles orthogonal to the same circle and coaxal each with a different pair of three others, all four being arbitrary, are coaxal with each other.

In the case of 2°, if PB_0 and PC_0, QC_0 and QA_0, RA_0 and RB_0 be the three pairs of tangents, real or imaginary, from the three points P, Q, R to the three pairs of original circles B_0 and C_0, C_0 and A_0, A_0 and B_0 respectively, and D, E, F the three centres of perspective of the latter at which the three lines QR, RP, PQ intersect collinearly with the three BC, CA, AB respectively (210, *a* and *b*); then, since by 134, *a*.,

$$\frac{QD}{RD}\cdot\frac{RE}{PE}\cdot\frac{PF}{QF}=1, \text{ and } \frac{BD}{CD}\cdot\frac{CE}{AE}\cdot\frac{AF}{BF}=1,$$

therefore, from the first, immediately,

$$\frac{QD.QR}{RD.RQ}\cdot\frac{RE.RP}{PE.PR}\cdot\frac{PF.PQ}{QF.QP}=-1;$$

but since from the three constant ratios of similitude of the three pairs of circles B_0 and C_0, C_0 and A_0, A_0 and B_0 respectively, by Euc. III. 35, 36,

$$\frac{QD.QR}{RD.RQ}=-\frac{BD}{CD}\cdot\frac{QC_0^2}{RB_0^2},$$

$$\frac{RE.RP}{PE.PR}=-\frac{CE}{AE}\cdot\frac{RA_0^2}{PC_0^2},$$

$$\frac{PF.PQ}{QF.QP}=-\frac{AF}{BF}\cdot\frac{PB_0^2}{QA_0^2},$$

therefore, from the second, by composition,

$$\frac{QC_0^2}{RB_0^2}\cdot\frac{RA_0^2}{PC_0^2}\cdot\frac{PB_0^2}{QA_0^2}=1;$$

from which, since by 192, Cor. 1°,

$$\frac{PB_0^2}{PC_0^2}=\frac{BX}{CX},\quad \frac{QC_0^2}{QA_0^2}=\frac{CY}{AY},\quad \frac{RA_0^2}{RB_0^2}=\frac{AZ}{BZ},$$

therefore &c. (134, *a*.)

Of the eight circles of contact of any system of three arbitrary circles, Dr. Hart has shewn, by a process, of the first part of which he has given an abstract in the *Quarterly Journal of Pure and Applied Mathematics*, Vol. IV. page 260, that they may always be divided, in four different ways, into two groups of four and their four conjugates, having each a fourth common circle of contact in addition to the original three; and Dr. Salmon, by an analysis of remarkable elegance, which he has

given in Vol. VI. page 67 of the same periodical, has verified Dr. Hart's results and extended them to the more general figures of which circles are particular cases. The methods employed by both geometers, however, involve principles beyond the limits of the present work; and a demonstration of the property by Elementary Geometry, within the domain of which it manifestly lies, has not, so far as the Author is aware, been yet given.

END OF VOL. I.

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